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A variational principle for actions on symmetric symplectic spaces

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Abstract

We present a definition of generating functions of canonical relations, which are real functions on symmetric symplectic spaces, discussing some conditions for the presence of caustics. We show how the actions compose by a neat geometrical formula and are connected to the hamiltonians via a geometrically simple variational principle which determines the classical trajectories, discussing the temporal evolution of such "extended hamiltonians" in terms of Hamilton–Jacobi-type equations. Simplest spaces are treated explicitly.

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1. Introduction

The concepts of momentum and position set together on an equal footing define what is known today as phase space [19], geometrically a differentiable manifold M endowed with a symplectic form ω (nondegeneracy could be relaxed [22,45], but here we keep to the symplectic case). This generalizes Hamilton's formalism, as presented in standard textbooks [1,2,11,36].

Usually, however, the phase space is first treated as an euclidean $2n$-space, or even a cotangent bundle, with canonical coordinates $\{(p_i, q_i)\}$. While Hamilton's equation can be generically written down for each real differentiable function on any symplectic manifold,

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other important and familiar features of the canonical formalism do not always extend in an obvious fashion.

Furthermore, although Darboux's theorem allows for a local canonical coordinatization of general symplectic manifolds, this is not always the most convenient one to use. It is therefore desirable to have a formalism which takes into account the specific geometry of the phase space, explicitly, as a further step of the "geometrical revenge" in dynamics [44].

Actually, the importance of considering nontrivial geometries can be seen from many different perspectives. Although the law of inertia sets euclidean geometry in a privileged status, once a system is subject to nontrivial constraints its proper phase space geometry ceases to be trivial, generically [9,12,22]. Similarly, for a system which is invariant under a group of symmetries, it is often possible to eliminate redundant degrees of freedom and the reduced phase space is also generally nontrivial [1,22,23]. Alternatively, not to mention relativistic contexts, one may be interested in the classical limit of quantum systems, which are noneuclidean, as spin systems [28,36]. Equivalently, it is convenient to have formalisms of classical dynamics intimately connected to some important representations used in quantum mechanics. For instance, coherent states [16,29] or, more pertinent to this work at hand, the "center representation" of operators (cartesian Weyl symbols) [26,27].

Again, it is desirable to have these "semiclassical" formulations in nontrivial geometries, but this requires previously well-defined classical formalisms.

Particularly, we need well-defined generating functions for finite canonical transformations. When M is the cotangent bundle over a configuration space Q, these functions naturally take their values on $Q \times Q$. When M is a Kähler manifold [8,17], whose metric and symplectic form are related via its complex structure, one can define generating functions which are bi-holomorphic on $M_{\mathbf{C}} \times \bar{M}_{\mathbf{C}}$, in which case the trajectories are complex.

Despite the utility of the complex and the configuration space actions in treating a variety of problems, we can greatly benefit from a formalism leading to actions which are real differentiable functions on nontrivial symplectic manifolds, in relationship to the real hamiltonian flows obtained from a real variational principle.

Here we present such a formalism. This work is a generalization on concepts developed for euclidean spaces [25,27], or for linearized neighborhoods of general symplectic manifolds [41], dating back to the original work of Poincaré [30]. The generating functions are real functions on M itself and not on a pair of local lagrangian coordinates. An argument of one such function, $m \in M$, can be viewed as the "center" of the canonical transformation $m_- \mapsto m_+$ which the function generates implicitly.

The corresponding variational principle states that, for an appropriate family of paths $\nu : [0, t] \to M$ whose endpoints are "centered" on m, the action

$$\left\{ -\int_\nu h(m'(t'), t')\,\mathrm{d}t' + \mathcal{S}_\nu \omega \right\} (m, t)$$

is stationary for a classical trajectory. Here, $\{\mathcal{S}_\nu \omega\}(m, t)$ is the symplectic area between the curve ν and the geodesic arc, centered on m, returning from $\nu(t)$ to $\nu(0)$.

The restriction on the full foregoing theory is that M be a symmetric symplectic space [13,17,20]. This means that M admits of a complete affine connection such that every point $m \in M$ is the isolated fixed point of an involutive symplectomorphism which coincides with geodesic inversion at m. Accordingly, the above formulation of the variational principle is invariant at least with respect to general transformations on M preserving the affine connection and the symplectic form.

The generalization from flat to nonflat symmetric symplectic spaces starts with the notion of double phase space, DM [27,44]. While in the flat case we define the reflection–translation group, from which the flat theory develops (Section 2), the equivalent construction for non-flat spaces is not enough and we must use the fact that DM is a symplectic groupoid [3,46].

Then, via a "symmetric exponential map", we view all structures of DM in (a subset of) the tangent bundle over M, TM. There we define the notion of central groupoid (Section 3). The properties of the pullback symplectic form allows us to see the graphs of canonical transformations as lagrangian submanifolds Λ of (subsets of) TM and define, in Section 4 (local), generating functions for finite canonical transformations (more generally, relations), which are real functions on M. Such functions generate well-defined canonical relations only when Λ is a graph over the zero section of TM. For canonical transformations, there is a further graphical condition. These considerations are discussed in Section 5.

The composition of two canonical transformations $\alpha_i : M \to M$, as generated by such "central actions", is treated in Section 6. Here, again we find that despite their abstract nature, the rule for composing them is very simple. If f_{α_i} is the central action for α_i, then

$$f_{\alpha_1} \triangle f_{\alpha_2}(m) = \mathrm{Stat}_{(m_1,m_2)}\{f_{\alpha_1}(m_1) + f_{\alpha_2}(m_2) + \Delta(m, m_1, m_2)\}$$

is the composed central action for $\alpha_2(\alpha_1)$, where $\Delta(m, m_1, m_2)$ is the symplectic area of the geodesic triangle with given midpoints. On general symmetric symplectic spaces, the importance of this function was first realized in the context of star product quantization [47], whose euclidean version has long been well established using the idea of centers and chords [4,27].

On the other hand, the above rule naturally generalizes the result previously obtained on $\mathbb{R}^{2n}$ [21,25]. By iterating these compositions, in Section 7, we arrive at the central variational principle by taking the limit of an infinite number of infinitesimal canonical transformations (Section 8).

Finally, we discuss the temporal evolution of such "finite-time extensions" of hamiltonians describing, in Section 9, a central version of the Hamilton–Jacobi equation and, more generally, their time derivative for any hamiltonian flow, mixing Hamilton–Jacobi with Poisson brackets.

While presenting these concepts we often make use of particular spaces to illustrate the text. We have focused on the simplest two-dimensional cases: the euclidean plane $\mathbb{R}^2$, the torus $\mathcal{T}^2$, the sphere S^2 and the noncompact hyperbolic plane H^2. Of course, extending these examples to their cartesian products $M = \mathbb{R}^{2n}$, $S^2 \times \cdots \times S^2$, $H^2 \times \mathcal{T}^2$, etc. is reasonably straightforward and has not been considered here. We emphasize, however, that the theory in principle applies to generic symmetric symplectic spaces of arbitrary (even) dimensions.

2. Central coordinates on flat spaces

Consider the euclidean plane $\mathbb{R}^2$, representing the very simplest phase space of a single degree of freedom. $\mathbb{R}^2$ is a group, under vector addition, and we can also identify the group product as a free transitive action of $\mathbb{R}^2$ on itself, the translations: $T_{\vec{\xi}}(\vec{x}) = \vec{x} +$

$\vec{\xi}$. Further, $\mathbb{R}^2$ admits a natural involution $\mathcal{R}_0(\vec{x}) = -\vec{x}$, which represents a reflection, or rotation by π, through the origin. Together with the identity transformation this gives an action of $\boldsymbol{Z}_2$ on $\mathbb{R}^2$. We can form the semidirect product: $RT := \boldsymbol{Z}_2 \ltimes \mathbb{R}^2$, which is called the reflection–translation group and can also be seen as a normal subgroup of the inhomogeneous symplectic group.

Then, $\mathcal{R}_{\vec{x}} = (T_{\vec{x}}\mathcal{R}_0 T_{-\vec{x}}) = T_{2\vec{x}}\mathcal{R}_0 = \mathcal{R}_0 T_{-2\vec{x}}$ is the element of RT which acts as reflection through the point $\vec{x} : \mathcal{R}_{\vec{x}}(\vec{x}') = 2\vec{x} - \vec{x}'$. The elements of $\{T_{\vec{\xi}}, \mathcal{R}_{\vec{x}}\}$ satisfy:

$$
\begin{aligned}
&T_{\vec{\xi}}T_{\vec{\xi}'} = T_{(\vec{\xi}+\vec{\xi}')}, \qquad T_{\vec{\xi}}\mathcal{R}_{\vec{x}} = \mathcal{R}_{(\vec{x}+\vec{\xi}/2)},\\
&\mathcal{R}_{\vec{x}}T_{\vec{\xi}} = \mathcal{R}_{(\vec{x}-\vec{\xi}/2)}, \qquad \mathcal{R}_{\vec{x}}\mathcal{R}_{\vec{x}'} = T_{2(\vec{x}-\vec{x}')}.
\end{aligned} \tag{2.1}
$$

Now, let $(\vec{x}_-, \vec{x}_+) \in \mathbb{R}^2 \times \mathbb{R}^2$ be such that $\vec{x}_+ = \mathcal{R}_{\vec{x}}(\vec{x}_-) \Leftrightarrow \vec{x}_- = \mathcal{R}_{\vec{x}}(\vec{x}_+)$. Then, $\vec{x} \in \mathbb{R}^2$ is called the center of the pair $(\vec{x}_-, \vec{x}_+)$. Actually, $\vec{x} = (1/2)(\vec{x}_- + \vec{x}_+)$, coinciding, for the euclidean metric, with the midpoint of the geodesic arc joining $\vec{x}_-$ to $\vec{x}_+$. Identifying $\vec{\xi} = \vec{x}_+ - \vec{x}_-$, the transformation $(\vec{x}_-, \vec{x}_+) \mapsto (\vec{x}, \vec{\xi})$ is a bijection.

Finally, we see how the group of translations on $\mathbb{R}^2$ can be interpreted as a composition of pairs: Let $\vec{x}_+ = T_{\vec{\xi}'}(\vec{x}_\lambda)$, $\vec{x}_- = T_{-\vec{\xi}''}(\vec{x}_\lambda)$. Then $\vec{x}_+ = T_{\vec{\xi}'}T_{\vec{\xi}''}(\vec{x}_-) = T_{(\vec{\xi}'+\vec{\xi}'')}(\vec{x}_-)$. But noticing that the action $\vec{x}_+ = T_{\vec{\xi}}(\vec{x}_-) = \vec{x}_- + \vec{\xi}$ can be identified by any of the sets $\{\vec{x}_-, \vec{\xi}\}$, $\{\vec{x}_+, \vec{\xi}\}$, $\{\vec{x}_-, \vec{x}_+\}$ uniquely, we can rewrite the composition of translations as:

$$
(\vec{x}_-, \vec{x}_\lambda) \odot (\vec{x}_\lambda, \vec{x}_+) = (\vec{x}_-, \vec{x}_+). \tag{2.2}
$$

Although (2.2) was derived from the group product on $\mathbb{R}^2$ it can be generalized for spaces which are not groups. In other words, we can turn the argument around and identify (2.2) as the fundamental algebraic structure on $M \times M$. This introduces the concept of groupoid.

3. Central groupoids

Definition 3.1. Let Γ, M be spaces. Γ is called a *groupoid* over M, denoted $\Gamma \rightrightarrows M$, if:

(Gd.0) $\exists$ two maps $P_-, P_+ : \Gamma \to M$, called the source and target maps, respectively.

(Gd.1) Let $(\Gamma \times \Gamma) \supset \Gamma_2 := \{(\gamma', \gamma'') | P_+(\gamma') = P_-(\gamma'')\}$. Γ_2 is called the set of composable elements. Then, $\exists$ an associative map "$\odot$" : $\Gamma_2 \to \Gamma$, $(\gamma', \gamma'') \to \gamma' \odot \gamma''$, called groupoid composition, or product, satisfying $P_-(\gamma' \odot \gamma'') = P_-(\gamma')$, $P_+(\gamma' \odot \gamma'') = P_+(\gamma'')$.

(Gd.2) $\exists$ an involution $i : \Gamma \to \Gamma$, called inversion, s.t. $\forall \gamma \in \Gamma$, its unique inverse $\bar{\gamma} \equiv i(\gamma)$ satisfies: $\bar{\gamma} \odot (\gamma \odot \gamma') = \gamma'$ if $(\gamma, \gamma') \in \Gamma_2$, $(\gamma' \odot \gamma) \odot \bar{\gamma} = \gamma'$ if $(\gamma', \gamma) \in \Gamma_2$.

M can be identified with the set of identities in Γ, $\Gamma_e := \{\gamma \odot \bar{\gamma}$, or $\bar{\gamma} \odot \gamma | \gamma \in \Gamma\}$. Every $\gamma_e \in \Gamma_e$ satisfies $P_-(\gamma_e) = P_+(\gamma_e)$, but the converse is not necessarily true.

Examples: The simplest examples of groupoids are:

(i) Γ is a group G, $\Gamma_e \equiv \{e\} \equiv M$, $\Gamma_2 \equiv G \times G$.

(ii) The pair groupoid $\Gamma \equiv M \times M$, P_- and P_+ being the first and second projections. In this case, $\Gamma_e \simeq M$ is the diagonal, inversion is permutation and the composition is

$$(m_-, m_\lambda) \odot (m_\lambda, m_+) = (m_-, m_+) \tag{3.1}$$

which generalizes (2.2). These two examples are complementary in the sense that they have, respectively, the minimal and the maximal identity spaces possible.

We now focus on those groupoids Γ which are also symplectic manifolds and for which groupoid composition respects its symplectic structure [3,46].

Definition 3.2. A groupoid $\Gamma \rightrightarrows M$ is *symplectic* if (Γ, ω_Γ) is a symplectic manifold and:

(LGd) $M \simeq \Gamma_e$ is a submanifold of Γ, $P_\pm$ are submersions and $i : \Gamma \to \Gamma, \odot : \Gamma_2 \to \Gamma$ are smooth. In this case Γ is called a Lie Groupoid.

(SGd) The graph of "$\odot$" is a lagrangian submanifold of $\Gamma \times \Gamma \times \bar{\Gamma}$, where $\Gamma \to \bar{\Gamma}$ is an antisymplectic isomorphism: $(\bar{\Gamma}, \omega_{\bar{\Gamma}}) \equiv (\Gamma, -\omega_\Gamma)$. In shorthand notation, we write this as an "additive" property: $\omega_\Gamma(\gamma_1 \odot \gamma_2) \approx \omega_\Gamma(\gamma_1) + \omega_\Gamma(\gamma_2)$.

As consequences, $M \simeq \Gamma_e$ is a lagrangian submanifold of Γ and inversion is anti-symplectic, i.e. the graph of "i" is a lagrangian submanifold of $\Gamma \times \Gamma$, $\omega_\Gamma(\bar{\gamma}) \approx -\omega_\Gamma(\gamma)$.

Example: Let M be a sympletic manifold, ω its symplectic structure. Then, the pair groupoid $\bar{M} \times M$, with sympletic structure $\omega_\Gamma = \delta\omega$, where

$$\delta\omega := P_+^*(\omega) - P_-^*(\omega) \equiv \omega_+ \ominus \omega_- \tag{3.2}$$

is a symplectic groupoid, called the *product* or *double phase space* and denoted *DM*. That is $DM \equiv (\bar{M} \times M) \rightrightarrows M$. Notice that while *DM* satisfies SGd, the cartesian product $M \times M$, with the usual symplectic form $\omega \oplus \omega$, does not.

In order to define the notion of "center" precisely, we need to consider phase spaces which are symplectic, but also have the property of being symmetric [13,17,20]. We now have the following definition.

Definition 3.3. A differentiable manifold M endowed with a symplectic form ω shall be called a *symmetric symplectic space* if (M, ω) admits of a complete affine connection ∇ which is symplectic, i.e. $\nabla\omega = 0$, such that $\forall m \in M$, m is the isolated fixed point of an involutive symplectomorphism $\mathcal{R}_m : M \to M$ which coincides with the inversion, at m, of the geodesic flow, for all geodesics through m. Accordingly, $\mathcal{R}_m$ satisfies $\mathcal{R}_m\mathcal{R}_{m'}\mathcal{R}_m = \mathcal{R}_{\mathcal{R}_m(m')}$, $\{\mathcal{R}_m\mathcal{R}_{m'}\}$ is the group of displacements on M and ∇ is the corresponding canonical connection with null torsion and parallel curvature.

This generalizes the reflection–translation structure on euclidean space.

An important particular case is when ∇ *is the Levi-Civitta connection associated to a complete riemannian metric* η. In this case, $\mathcal{R}_m^*(\eta, \omega) = (\eta, \omega)$ and one can show [15,34] that (M, ω, η) is a *hermitian symmetric space*, that is, M is a Kähler manifold [8,17] whose complex structure J satisfies $\nabla J = 0$ and $\eta(JX, Y) \equiv \omega(X, Y), \forall X, Y \in \mathcal{X}(M)$, and for which the involution $\mathcal{R}_m$ is holomorphic.

More generally, part of the following theory, namely the local definition of central generating functions (Section 4), can be developed on general symplectic manifolds [30,40–42].

However, the geometrical law of composition (Section 6) and its consequences need the full setting of a symmetric symplectic space.

As simplest examples of these, we shall consider the flat euclidean plane $\mathbb{R}^2$ and torus $\mathcal{T}^2$, both being also groups, and as simplest nonflat examples, the sphere S^2 and the noncompact hyperbolic plane H^2, both being hermitian symmetric spaces.

When M is a symmetric symplectic space, let us denote the exponential map by $\exp_m : T_m M \to M, \vec{v} \mapsto \rho_\tau(1)$, where $\rho_\tau(t)$ is the geodesic in M defined by $\rho_\tau(0) = m, \dot{\rho}_\tau(0) = \vec{v}$. We can define a (restricted) groupoid structure directly on (a subset of) its tangent bundle TM, associated to DM via the *symmetric exponential map*:

$$\Phi : TM \to DM, \qquad \tau = (m, \vec{v}) \mapsto \gamma = (\exp_m(-\vec{v}), \exp_m(\vec{v})).$$

Here, Φ is "symmetric" in the sense that $(\mathcal{R}_m \circ \Phi)(m, \vec{v}) = \Phi(m, -\vec{v})$ and thus $\gamma \to \bar{\gamma} \equiv i(\gamma)$.

Denoting $\Omega = \Phi^*(\delta\omega)$, $\delta\omega$ defined in (3.2), we have that Ω is a closed 2-form on TM since $\delta\omega$ is a symplectic form on DM. If M has no closed geodesics, as for a hermitian symmetric space of noncompact type (or $\mathbb{R}^{2n}$), then Φ is a bijection and (TM, Ω) is a sympletic manifold.

Otherwise, Φ is not globally invertible. Consider the set $(TM)_0 \subset TM$, $(TM)_0$ being the maximal subset, connected to the zero section $T^0 M \equiv \{(m, \vec{0})\} \subset TM$, on which Φ is invertible. Denote $\Phi_0 := \Phi|_{(TM)_0}$, $\Omega_0 := \Omega|_{(TM)_0} \equiv \Phi_0^*(\delta\omega)$. Then, $((TM)_0, \Omega_0)$ is a symplectic manifold.

It is known [18] that for complete riemannian manifolds, only the stratum of $\Sigma_1 \subset TM$ has maximal dimension, where $\sigma_i(m) = \Sigma_i \cap T_m M$ is the subset of $T_m M$ defined by $I(v) = i$, $i = 1, 2, \ldots$ with $I(v)$ = number of geodesics of length $\|v\|$ which connect m and $\exp_m(v)$.

Thus, if M is a hermitian symmetric space of compact type (or the torus), although $\Phi_0 : (TM)_0 \to DM$ is not onto, it only leaves out a set of measure zero with respect to the Liouville form on DM obtained from $\delta\omega$. We conjecture that this may be true for other symmetric symplectic spaces with closed geodesics, as well. These facts motivate a slight modification in the concept of symplectic groupoids, suitable to our purposes. We now have the following definition.

Definition 3.4. Let M be a symmetric symplectic space. $((TM)_0, \Omega_0)$ shall be called the *standard central groupoid over* M, denoted $(TM)_0 \Rightarrow M$, satisfying:

(CG.0) $\exists$ three maps $P_0, P_-, P_+ : (TM)_0 \to M$, called the central, source and target maps, respectively, where $P_0(m, \vec{v}) = m$ is the natural projection and $P_\pm(m, \vec{v}) = \exp_m(\pm\vec{v})$.

(CG.1) On the restricted set of composable elements $((TM)_0 \times (TM)_0) \supset (TM)_0^2 := \{(\tau', \tau'') | P_+(\tau') = P_-(\tau'')$ and $(P_-(\tau'), P_+(\tau'')) \in \mathrm{Image}(\Phi_0) \subset DM\}$, the composition "$\odot$" : $(TM)_0^2 \to (TM)_0$, satisfying (Gd.1), is defined by $\tau' \odot \tau'' \equiv \Phi_0^{-1}(P_-(\tau'), P_+(\tau''))$.

(CG.2) $\exists$ an involution $i : (TM)_0 \to (TM)_0$, called inversion, satisfying all properties in (Gd.2) plus $P_0(i(\tau)) = P_0(\tau)$, $\forall \tau \in (TM)_0$. Specifically, if $\tau = (m, \vec{v})$ then $\bar{\tau} \equiv i(\tau) = (m, -\vec{v})$.

(CG.3) The identity space is the zero section and inversion is smooth. However, $P_\pm$ are only locally submersions and $\odot$ is locally smooth. $(TM)_0$ is a local Lie groupoid.

(CG.4) The graph of "$\odot$" is the union of lagrangian submanifolds of $(TM)_0 \times (TM)_0 \times (\overline{TM})_0$. The graph of "$i$" is a lagrangian submanifold of $(TM)_0 \times (TM)_0$, the zero section being lagrangian in $(TM)_0$. We may call $((TM)_0, \Omega_0)$ a local symplectic groupoid.

When M has no closed geodesics, $(TM)_0 \equiv TM$, $\Phi_0 \equiv \Phi$, TM is diffeomorphic to DM. In this case, the words "restricted" and "local" do not apply and the central groupoid is a bonafide symplectic groupoid with the extra structure of a central map.

Examples: $(T\mathbb{R}^2)_0 \equiv T\mathbb{R}^2$, $(TH^2)_0 \equiv TH^2$, the trivial cases since both $\mathbb{R}^2$ and H^2 have no closed geodesics. For $\mathcal{T}^2$ we have the following: Let AS^1 denote the set of antipodals in $S^1 \times S^1$, i.e. $\{(\varphi, \varphi \pm \pi)\}$, under the usual representation $S^1 \ni e^{i\varphi}, \varphi \in [0, 2\pi]$. The subset $T\mathcal{T}^2 \supset (T\mathcal{T}^2)_0 := \{\tau = (p, q; v_p, v_q) | |v_p|, |v_q| < \pi/2\}$ is s.t. $\Phi_0 : (T\mathcal{T}^2)_0 \to D\mathcal{T}^2 \backslash (AS^1 \times AS^1)$ is bijective. Clearly $AS^1 \times AS^1$ is a set of measure zero in $\mathcal{T}^2 \times \mathcal{T}^2$.

For S^2 the situation is very similar: Let AS^2 denote the set of antipodals in DS^2. Taking the subset $TS^2 \supset (TS^2)_0 := \{\tau = (m, \vec{v}) | |\vec{v}| < \pi/2\}$, then $\Phi_0 : (TS^2)_0 \to DS^2 \backslash AS^2$ is bijective. Again, AS^2 has measure zero in DS^2.

However, when M has closed geodesics, the standard central groupoid is not the only central groupoid possible. To see that, consider another subset $(TM)_1 \subset TM$ which satisfies: $(TM)_1 \cap (TM)_0 = \emptyset$ and $\Phi((TM)_1) = \Phi((TM)_0) \equiv (DM)_0 \subset DM$. Furthermore, denoting the map restriction by $\Phi_1 \equiv \Phi|_{(TM)_1}$, we limit attention to those subsets on which Φ_1 is almost everywhere injective. In other words, the set $N \subset (DM)_0$ on which Φ_1^{-1} is multiple valued has measure zero with respect to the Liouville form on DM obtained from $\delta\omega$.

Then, defining an equivalence relation among different pre-images in $(TM)_1$ of the same point in $(DM)_0$, we have another bijection $\tilde{\Phi}_1 : (\widetilde{TM})_1 \to (DM)_0$, where $(\widetilde{TM})_1 \equiv (TM)_1/\sim_1$ with the equivalence relation $\tau \sim_1 \tau'$ iff $\Phi_1(\tau) = \Phi_1(\tau')$. Accordingly, we denote such points in $(\widetilde{TM})_1$ by $\tilde{\tau} = [\tau]_1$, where $[\tau]_1 = \{\tau' \in (TM)_1 | \tau' \sim_1 \tau\}$. If $M \subset (DM)_0$ is the diagonal, its pre-image $\tilde{\Phi}_1^{-1}(M) := \widetilde{T^1M}$ is a global cross section of $(\widetilde{TM})_1$ setting a fibration $P_1 : (\widetilde{TM})_1 \to \widetilde{T^1M}$ s.t., if $\tilde{\tau} \in (\widetilde{TM})_1$ and $\tilde{\tau}^1 \in \widetilde{T^1M}$ then $P_1(\tilde{\tau}) = \tilde{\tau}^1$ iff $P_0(\tau) = P_0(\tau^1)$. In other words, the central map P_1 is essentially $P_0 : TM \to M$ itself. We denote $\Omega_1 \equiv \tilde{\Phi}_1^*(\delta\omega)$.

Definition 3.5. We call $((\widetilde{TM})_1, \Omega_1) \xrightarrow{P_1} (\widetilde{T^1M}, \omega)$ a *nonstandard central groupoid*. On $((\widetilde{TM})_1 \times (\widetilde{TM})_1) \supset (\widetilde{TM})_1^2 := \{(\tilde{\tau}, \tilde{\tau}') | \tilde{P}_+(\tilde{\tau}) = \tilde{P}_-(\tilde{\tau}') \text{ and } (\tilde{P}_-(\tilde{\tau}), \tilde{P}_+(\tau')) \in (DM)_0\}$, where $\tilde{P}_\pm(\tilde{\tau}) \equiv P_\pm(\tau), \forall \tau \in \tilde{\tau}$, the restricted groupoid product $\odot : (\widetilde{TM})_1^2 \to (\widetilde{TM})_1$ is given by $\tilde{\tau} \odot \tilde{\tau}' \equiv \tilde{\Phi}_1^{-1}(\tilde{\Phi}_1(\tilde{\tau}) \odot \tilde{\Phi}_1(\tilde{\tau}'))$, and inversion by $i(\tilde{\tau}) \equiv \tilde{\Phi}_1^{-1}(i(\tilde{\Phi}_1(\tilde{\tau})))$. If $(\widetilde{TM})_2 \xrightarrow{P_2} \widetilde{T^2M}$ is another central groupoid, it is *strongly equivalent* to $(\widetilde{TM})_1$ if, for every $\tilde{\tau} \in (\widetilde{TM})_1$ and $\tilde{\tau}' \in (\widetilde{TM})_2$ s.t. $\tilde{\Phi}_1(\tilde{\tau}) = \tilde{\Phi}_2(\tilde{\tau}')$, it follows $P_1(\tilde{\tau}) = P_2(\tilde{\tau}')$. If $\tilde{\Phi}_1(\tilde{\tau}) = \tilde{\Phi}_2(\tilde{\tau}')$ but $P_1(\tilde{\tau}) \neq P_2(\tilde{\tau}')$, $\tilde{\tau}'$ is *weakly equivalent* to $\tilde{\tau}$.

Examples: When $M = \mathcal{T}^2$, there are infinite central groupoids, but only four strongly inequivalent ones: $(T\mathcal{T}^2)_{01} \equiv \{\tau = (p, q, v_p, v_q) || v_p| < \pi/2, |v_q - \pi| < \pi/2\}$, similarly for $(T\mathcal{T}^2)_{10}$, where $|v_p - \pi| < \pi/2$, $|v_q| < \pi/2$, and $(T\mathcal{T}^2)_{11}$, where $|v_p - \pi|, |v_q - \pi| < \pi/2$, together with the standard one $(T\mathcal{T}^2)_0 \equiv (T\mathcal{T}^2)_{00}$. In this case, all the nonstandard central groupoids are simple subsets of *TM*, i.e. no equivalence relations had to be considered since each $\Phi_{ij} \equiv \Phi|_{(T\mathcal{T}^2)_{ij}}$ is already injective. Each point in $(D\mathcal{T}^2)_0$ has four strongly inequivalent pre-images: (p, q, v_q, v_q), $(p \pm \pi, q, v_p \pm \pi, v_q)$, $(p, q \pm \pi, v_p, v_q \pm \pi)$ and $(p \pm \pi, q \pm \pi, v_p \pm \pi, v_q \pm \pi)$.

In the case of S^2, there are only two strongly inequivalent central groupoids. The standard one and a nonstandard central groupoid: $\widetilde{(TS^2)}_1 \equiv (TS^2)_1/\sim_1$, where $(TS^2)_1 \equiv \{(m, \vec{v})||\vec{v}| \in (\pi/2, \pi]\}$. The equivalence relation is nontrivial only when $|\vec{v}| = \pi$, coinciding with the definition of the nonstandard pre-image of the diagonal, $\widetilde{T^1S^2} := \{[\tau^1]_1$, where $\tau^1 = (m, \vec{v}_1) \in (TS^2)_1$ is s.t. $|\vec{v}_1| = \pi$ and $(m, \vec{v}_1) \sim_1 (m', \vec{v}_1')$ iff $m = m'\}$. Thus, each point on $(DS^2)_0$ has two strongly inequivalent pre-images. If we denote $\Phi_0^{-1}(m_-, m_+) = (m, \vec{v}) \equiv (\theta, \varphi; v, \beta)$, see below for definition of the fiber coordinates (v, β), then $\tilde{\Phi}_1^{-1}(m_-, m_+) = (\pi - \theta, \varphi - \pi; \pi - v, \beta - \pi)$.

The reason for considering such nonstandard pre-images in *TM* is, of course, that when M has closed geodesics, the geodesic arc connecting two points in M is not unique. Accordingly, the standard central groupoid refers to the shortest of such geodesic arcs and the k-nonstandard central groupoid refers to the k-fold ones. Here, a k-fold geodesic from m' to m'' is defined as the geodesic $(m' \to m) * L_k(m) * (m \to m'')$, where $L_k(m)$ is a k-fold geodesic loop based at m, the midpoint of the short geodesic, and $*$ denotes free product. (If $k \equiv 0$, $L_0 \equiv id$ is the trivial loop and $(m' \to m'')$ is the short geodesic.) For instances of nontrivial geodesic loops, on $\mathcal{T}^2$, $L_k \equiv L_{(a,b)} \equiv L_p^a * L_q^b$, where L_p, L_q are the single irreducible circuits, while on S^2, L_k is a $|k|$-repetition of a geodesic meridian and it is easy to see that in this case we only need to consider strongly inequivalent geodesics. We also consider the following definition.

Definition 3.6. *Let* $(\widetilde{TM})^2_{ij} := \{(\tilde{\tau}, \tilde{\tau}') \in (\widetilde{TM})_i \times (\widetilde{TM})_j | \tilde{P}_+(\tilde{\tau}) = \tilde{P}_-(\tilde{\tau}')$ and $(\tilde{P}_-(\tilde{\tau}), \tilde{P}_+(\tilde{\tau}')) \in (DM)_0\}$. We define generalized, or *mixed compositions* (which are still restricted in the original sense) as $c^k_{ij} : (\widetilde{TM})^2_{ij} \to (\widetilde{TM})_k$, by $c^k_{ij}(\tilde{\tau}, \tilde{\tau}') \equiv \tilde{\Phi}_k^{-1}(\tilde{\Phi}_i(\tilde{\tau}) \odot \tilde{\Phi}_j(\tilde{\tau}'))$. If M is not simply connected, we may want to consider only mixed compositions whose geodesic triangles of composition are reducible circuits, in which case the triple $(i, j; k)$ is called *reducible*. When M is simply connected, it is enough to consider mixed compositions within the set of strongly inequivalent central groupoids.

Now, the application of central groupoids in classical dynamics rests on the following:

Definition 3.7. Let N be a symplectic space. A *polarization* on N is an integrable foliation of N by lagrangian leaves.

Lemma 3.1. *P_0 sets a polarization on $((TM)_0, \Omega_0)$, called the vertical or central polarization.*

Proof. First, every fibration is an integrable foliation. Second, since each fiber is (an open subset of) the tangent space at each point in M, the fiber dimension is half the total dimension of $(TM)_0$. To prove isotropy with respect to $\Omega_0 = \Phi_0^*(\delta\omega)$, note that inversion is anti-symplectic in $(TM)_0$, i.e. $i^*(\Omega_0) = -\Omega_0$. But, denoting the fiber restriction by $(T_m M)_0$, we have that $i_m := i|_{(T_m M)_0} \equiv \Phi_0^{-1} \circ \mathcal{R}_m \circ \Phi_0$. Since ω is $\mathcal{R}_m$ invariant, i.e. $\mathcal{R}_m^*(\omega) = \omega$, it follows that $i_m^*((\Omega_0)_m) = (\Omega_0)_m$, where $(\Omega_0)_m \equiv \Omega_0|_{(T_m M)_0}$. Thus, $(\Omega_0)_m \equiv 0$. □

Further, we realize that Lemma 3.1 applies to every $P_i : (\widetilde{TM})_i \to \widetilde{T^i M}$, as well.

Thus, in what follows we should consider all possible central groupoids similarly. However, we shall mostly deal with the standard central groupoid, both to simplify the treatment and because we will often focus on the limit $|\vec{v}| \to 0$, which can only take place in $(TM)_0$. Accordingly, we often drop the denomination "standard" in following definitions and discussions.

Finally, we should compare the central groupoids with the local isomorphism $DM \to T^*M$, in a neighborhood of the diagonal in DM, for any symplectic manifold M [40–42]. The differences lie in the scope and properties of the symplectic structure. While $((\widetilde{TM})_i, \Omega_i)$ contains explicit information on the affine geometry (geodesic structure) of M, the canonical symplectic form on T^*M does not.

Formulae: To base the abstract concepts of this paper, we will often illustrate them in the simplest spaces $M = \mathbb{R}^2, \mathcal{T}^2, S^2, H^2$. We now provide some useful local formulae for their standard central groupoids. Start with the flat spaces, locally identical. Take coordinates on $\mathbb{R}^2$ as canonical pairs, with usual metric and symplectic form. The tangent bundle is also a linear space: $T\mathbb{R}^2 \ni \vec{\tau} = (\vec{x}; \vec{v}) = (p, q; v_p, v_q)$, and the symmetric exponential map is

$$\Phi(\vec{\tau}) \equiv \Phi(\vec{x}; \vec{v}) = (\vec{x} - \vec{v}; \vec{x} + \vec{v}) \equiv (\vec{x}_-, \vec{x}_+) \tag{3.3}$$

with pull-back symplectic form given by

$$\Omega = 2\,\mathrm{d}\Sigma, \qquad \Sigma = v_p\,\mathrm{d}q - v_q\,\mathrm{d}p. \tag{3.4}$$

For the sphere, take local polar coordinates: $S^2 \ni m = (\theta, \varphi), \theta \in [0, \pi], \varphi \in [0, 2\pi]$. We shall be using the following abbreviations:

$$S_\alpha \equiv \sin(\alpha), \qquad C_\alpha \equiv \cos(\alpha), \qquad T_\alpha \equiv \tan(\alpha) \quad \text{with } S^{-1}(f), C^{-1}(f), T^{-1}(f)$$

denoting their respective inverses. With usual metric and symplectic form, the natural coordinates on the tangent bundle are $TS^2 \ni \tau = (\theta, \varphi; \dot{\theta}, \dot{\varphi})$, but it is more convenient to introduce polar coordinates on the fibers as well, $\tau = (\theta, \varphi; v, \beta)$ where

$$v = |\vec{v}| = \sqrt{\dot{\theta}^2 + S_\theta^2 \dot{\varphi}^2}, \qquad vC_\beta = \dot{\theta}, \qquad vS_\beta = S_\theta \dot{\varphi}.$$

With these local coordinates we write the symmetric exponential map as

$$\Phi_0(\theta, \varphi; v, \beta) = (\theta_-, \varphi_-; \theta_+, \varphi_+), \qquad \theta_\pm = C^{-1}(C_\theta C_v \mp S_\theta S_v C_\beta),$$
$$\varphi_\pm = \varphi \pm T^{-1}\left(\frac{S_v S_\beta}{g_\pm}\right) + (1 - \mathrm{Sign}\,(g_\pm))\frac{\pi}{2}, \qquad g_\pm = S_\theta C_v \pm C_\theta S_v C_\beta \tag{3.5}$$

and the pull-back symplectic form $\Omega_0 \equiv \Phi_0^*(\delta\omega)$ is given by

$$\Omega_0 = 2\,\mathrm{d}\Sigma, \quad \Sigma = S_v(C_\beta S_\theta\,\mathrm{d}\varphi - S_\beta\,\mathrm{d}\theta). \tag{3.6}$$

On H^2, we adapt the local spherical formulas by letting

$$\theta \mapsto \mathrm{i}\rho, \qquad v \mapsto \mathrm{i}\mu, \tag{3.7}$$

where $\mathrm{i} = \sqrt{-1}$. Thus, $S_\theta \mapsto \mathrm{i}\tilde{S}_\rho$, $C_\theta \mapsto \tilde{C}_\rho$, $S_v \mapsto \mathrm{i}\tilde{S}_\mu$, $C_v \mapsto \tilde{C}_\mu$, with

$$\tilde{S}_\alpha \equiv \sinh(\alpha), \quad \tilde{C}_\alpha \equiv \cosh(\alpha), \quad \tilde{T}_\alpha \equiv \tanh(\alpha) \quad \text{and}$$
$$\tilde{S}^{-1}(f), \quad \tilde{C}^{-1}(f), \quad \tilde{T}^{-1}(f)$$

the respective inverses. Then, from (3.5) and (3.6) we get Φ and $-\Omega$ for H^2.

We can see explicitly from (3.4), (3.6) and (3.7) that the vertical spaces are isotropic in these examples. The same holding for the zero section $T^0M \simeq M$. Notice also that the pull-back symplectic form is the exact derivative of a symplectic potential without any vertical differential components. This fact shall be thoroughly exploited in what follows.

4. The central equation

We started by emphasizing the algebraic structure on double phase spaces, or on central groupoids. Historically, the symplectic structure was predominant, however, for it introduced the very useful concept of action, or generating function of a canonical transformation on the original, or simple phase space (M, ω). We now recall the concept of generating function, adapting it to our context.

Let α be such a symplectomorphism $M \to M$, $\alpha^*(w) = w$. Its graph $\mathcal{L}_\alpha$ in the double phase space, $DM \supset \mathcal{L}_\alpha := \{(m_-, m_+)|m_+ = \alpha(m_-)\}$, is a lagrangian submanifold for the symplectic form $\delta\omega$, i.e. $\delta\omega|\mathcal{L}_\alpha \equiv 0$. Similarly, if $\mathcal{L}_\alpha \subset \mathrm{Image}\,(\Phi_0) \equiv (DM)_0 \subset DM$, then its pre-image $\Lambda_\alpha := \Phi_0^{-1}(\mathcal{L}_\alpha)$ is lagrangian in the central groupoid $((TM)_0, \Omega_0)$.

Conversely, every lagrangian submanifold $\Lambda_\alpha := \Phi_0^{-1}(\mathcal{L}_\alpha)$ defines a *symplectic* or *canonical relation* on M [42,44], which is a canonical transformation when $\mathcal{L}_\alpha$ is a graph over $M_- \subset DM$.

Generically, we consider those subsets $\mathcal{L}_\alpha^{(r)} \subset \mathcal{L}_\alpha$, satisfying $\mathcal{L}_\alpha^{(r)} \subset \mathrm{Image}\,(\Phi_0)$, and their corresponding pre-images $\Lambda_\alpha^{(r)}$ in $(TM)_0$.

The definition of local *generating functions* for lagrangian submanifolds amounts to realizing such manifolds, locally, as graphs of their corresponding generating functions.

This depends on the choice of a local *symplectic potential* (a local 1-form whose derivative is the symplectic form) suitable to a polarization which is, at least locally, a fibration over a referential lagrangian submanifold containing the supports of those generating functions [37,40–42,44].

For the central groupoid over M, we take M itself, seen as the zero section, as the referential lagrangian submanifold for the central polarization. The suitable symplectic potential is given in the following definition.

Definition 4.1. Let $((TM)_0, \Omega_0)$ be the standard central groupoid over (M, ω). A symplectic potential Z_0, for Ω_0, shall be called a standard *central potential* if it satisfies

$$X \rfloor Z_0 = P_0'(X) \rfloor Z_0, \quad \forall X \in \mathcal{X}((TM)_0). \tag{4.1}$$

Here, P_0' is the differential of P_0 and $\vec{x} \rfloor \alpha$ denotes the vector-form contraction.

Of course, any $Z = Z_0 + \mathrm{d}Q$, $Q \in C^k_{\mathbb{R}}((TM)_0)$, is another potential, but not generally central. Condition (4.1) tells us that central potentials have no vertical differential components, so we can identify these potentials explicitly in our examples as $Z_0 = 2\Sigma$, from (3.4), (3.6) and (3.7).

In these particular examples, Z_0 is a global potential on $(TM)_0$, i.e., $\Omega_0 \equiv \mathrm{d}Z_0$ is exact. We argue that the general case follow these known examples.

First, such a potential always exists in a small neighborhood of the zero section T^0M, just take the exact 2-form $\dot{\omega}$ on TM defined by $\dot{\omega}(m, \vec{v}) := \mathrm{d}(\vec{v} \rfloor \omega)$, $\forall \vec{v} \in \mathcal{X}(M)$. Then, $\dot{\omega} \equiv L_{\vec{v}}(w)$, the Lie derivative of ω, since ω is closed. On the other hand, when $|\vec{v}| \equiv t \tilde{\to} 0$

$$\Omega_0(m, \vec{v}) \tilde{\to} 2 \lim_{t\to 0} \left\{ \left(\frac{1}{t}\right) \int_0^t L_{\vec{v}}(\omega)\, \mathrm{d}t' \right\} \simeq 2L_{\vec{v}}(\omega),$$

that is, $\Omega_0 \tilde{\to} 2\dot{\omega} = 2\,\mathrm{d}(\vec{v} \rfloor \omega)$. Thus, in a neighborhood of T^0M, $\Omega_0 \tilde{\to} 2\,\mathrm{d}\dot{\zeta}$, where $\dot{\zeta}(m, \vec{v}) := \vec{v} \rfloor \omega(m)$ clearly satisfies (4.1). (On the flat examples, $Z_0 \equiv 2\dot{\zeta}$ globally, but generally $2\dot{\zeta}$ only approximates the central potential in a small neighborhood of T^0M.) Finally, T^0M is lagrangian for Ω_0 and each vertical fiber is also lagrangian, as well as contractible. Hence, $\Omega_0 \equiv \mathrm{d}Z_0$ is exact.

Recalling that $(1/2)Z_0(m, \vec{v}) \tilde{\to} \dot{\zeta}(m, \vec{v}) \equiv \vec{v} \rfloor \omega$, as $|\vec{v}| \to 0$, we may refer to the 1-form $\dot{\zeta}$ as *Hamilton's potential*, since it is intimately connected to *Hamilton's equation*.

To see this, remember that we can write the latter as a map $\dot{z}[h] : M \to TM$, $\forall h \in C^k_{\mathbb{R}}(M)$, by $m \mapsto \tau_h$, where $\tau_h = (m, \vec{v}_h)$ and $\vec{v}_h$ is given by $(\mathrm{d}h + \vec{v}_h \rfloor \omega)(m) = 0$, or equivalently, $\dot{\zeta}(\tau_h) = -\mathrm{d}h(m)$, $\forall m \in M$. Or still, by denoting $TM \supset \dot{\Lambda}_h :=$ graph of $\dot{z}[h]$, Hamilton's equation becomes $\dot{\zeta}|_{\dot{\Lambda}_h} = -\mathrm{d}h$, implicitly defining $\dot{z}[h]$.

Similarly, *the existence of a central potential for Ω_0 allows for a "finite time" extension of Hamilton's formalism*, now in the context of generating functions. Thus, suppose that a lagrangian submanifold $\Lambda_\alpha \subset (TM)_0$ is locally a graph over $T^0M \simeq M$, that is,

$$\operatorname{Rank}(P_0'|T_\tau \Lambda_\alpha) = 2n = \dim(M), \quad \forall \tau \in \Lambda_\alpha \text{ s.t. } P_0(\tau) = m \in U \subset M \tag{4.2}$$

then, since $\mathrm{d}Z_0|\Lambda_\alpha = 0$, from (4.1) we obtain the following proposition.

Proposition 4.1. *For every lagrangian submanifold $\Lambda_\alpha \subset (TM)_0$ satisfying (4.2) there exists a standard central generating function $f_\alpha \in C^k_{\mathbb{R}}(M)$, satisfying the standard central equation*

$$Z_0|_{\Lambda_\alpha} = \mathrm{d} f_\alpha \quad on\ (TU)_0 \subset (TM)_0. \tag{4.3}$$

Conversely, for a given f_α, the above equation defines Λ_α implicitly, i.e. it provides a standard section $z[f_\alpha] \equiv F_\alpha : U \to (TU)_0$ which is well defined when condition (4.2) is satisfied and as long as f_α satisfies appropriate standard consistency conditions.

We shall see examples of such consistency conditions shortly. Now, the map F_α can be multiple valued if $(T_m M)_0 \cap \Lambda_\alpha$ is not unique, in which case we should break Λ_α into branches $\Lambda_\alpha^{(r)}$, each one uniquely given by a map $F_\alpha^{(r)}$ in a subset of M via $Z_0|_{\Lambda_\alpha^{(r)}} = \mathrm{d} f_\alpha^{(r)}$, for each $f_\alpha^{(r)}$, except for $\{\tau_k\} \subset \Lambda_\alpha$ where $\mathrm{Rank}(P_0'|T_{\tau_k}\Lambda_\alpha) < 2n$. Then, via the symmetric exponential map, (4.3) locally generates a lagrangian submanifold $\mathcal{L}_\alpha \subset DM$, a canonical relation on M. The similarity between Hamilton's equation and the central equation is striking.

However, the former provides *infinitesimal* transformations while the latter generates *finite* relations. For this reason, not every canonical relation can be generated by a real function on M via the central equation, everywhere. Generically, the presence of *central catastrophes*, $\{\tau_k\} \subset \Lambda_\alpha$ s.t. $\mathrm{Rank}(P_0'|T_{\tau_k}\Lambda_\alpha) < 2n$, is unavoidable. Their projections, $\{m_k\} \equiv \{P_0(\tau_k)\} \subset M$, are called *central caustics*. To circumvent this problem, new sets of "complementary" generating functions are needed [25]. Alternatively, we can let the functions depend on extra parameters and look for their stationary points [6,7,39].

On the other hand, by introducing a real parameter $\lambda = t/2$ (a scale) in the map $\dot{z}[h]$, for instance, by multiplying every hamiltonian h by $t/2$, we can see Hamilton's equation as a map from M into a small neighborhood of the zero section in TM, if t is sufficiently small. Since in such a neighborhood $(1/2)Z_0$ and $\dot{\zeta}$ are approximately equal, $f_\alpha = -th$ is a central generating function for the infinitesimal canonical transformation generated by h via Hamilton's equation. Since these are always well defined, every infinitesimal canonical transformation can be generated by a central function. This is obvious if we notice that such transformations are small deformations of the identity, associated to lagrangian submanifolds in $(TM)_0$ which are small deformations of the zero section, thus satisfying (4.2).

Furthermore, the map $F_\alpha = -tH : M \to (TM)_0$, obtained via the central equation from function $f_\alpha = -th$, where h is the hamiltonian, is given by $m \mapsto (m, (\varepsilon/2)\vec{v}_h(m))$, for sufficiently short times $t \simeq \varepsilon$. Since $(\varepsilon/2)\vec{v}_h \to 0$, as $\varepsilon \to 0$, for very short times $t \simeq \varepsilon \to 0$, the implicit canonical transformation (m_-, m_+) obtained via the symmetric exponential map coincides with the linearized version $(\vec{x} - (\varepsilon/2)\vec{v}_h, \vec{x} + (\varepsilon/2)\vec{v}_h)$, for any choice of local linear coordinates on $U \subset M$, s.t. $m \simeq \vec{x}$, regardless of the specific affine geometry of M.

Thus, for very short time motion, the geodesic segment centered on m converges onto the hamiltonian orbit that propagates from m forwards and backwards in time.

Now, we remark that generating functions are traditionally defined on the double phase space DM, but this usually requires us to previously take polarizations on M itself.

For general cotangent bundles, with vertical polarization, the generating functions naturally take their values from pairs of base space points, $f \equiv f(q_-, q_+)$.

For general Kähler manifolds, with complex polarizations, the natural generating functions are bi-holomorphic functions, $f \equiv f(z_-, z_+)$, $z_\pm$ complex.

The present approach allows us to consider real generating functions on a general symmetric symplectic space M itself, defined via a real polarization on the central groupoid. These functions can sometimes be pictured as "finite time hamiltonians".

More generally, such central generating functions can locally be defined on any symplectic manifold by considering a local polarization of DM which is transversal to the diagonal $M \subset DM$. If linear coordinates are chosen on a neighborhood of a point in M, these functions can be mapped to Poincaré's generating functions [30,41]. However, the full geometrical properties of the central generating functions need the symmetric symplectic setting.

Also, we must point out that other versions of the central equation are available from the following proposition.

Proposition 4.1'. *If M has closed geodesics, there are nonstandard central potentials Z_i s.t.*

$$\mathrm{d}Z_i = \tilde{\Phi}_i^*(\delta\omega) \quad \text{and} \quad X \rfloor Z_i = P_i'(X) \rfloor Z_i, \quad \forall X \in \mathcal{X}((\widetilde{TM})_i). \tag{4.1'}$$

Each lagrangian submanifold $\Lambda_\alpha^i = \tilde{\Phi}_i^{-1}(\mathcal{L}_\alpha) \subset (\widetilde{TM})_i$, satisfying

$$\mathrm{Rank}(P_i'|T_{\tilde{\tau}}\Lambda_\alpha^i) = 2n, \qquad \text{over } U \subset M \simeq \widetilde{T^i M} \tag{4.2'}$$

is generated by a nonstandard central function f_α^i via a nonstandard central equation:

$$Z_i|_{\Lambda_\alpha^i} = \mathrm{d}f_\alpha^i \quad \textit{on } (\widetilde{TU})_i \subset (\widetilde{TM})_i, \tag{4.3'}$$

which provides a nonstandard section $F_\alpha^i : U \to (\widetilde{TU})_i$ defining Λ_α^i. When no distinction is made, or by means of generalization, we refer to (4.3) or (4.3') simply as the central equation.

Notice that we have abbreviated the standard notation: $f_\alpha^0 \equiv f_\alpha$, $\Lambda_\alpha^0 \equiv \Lambda_\alpha$, $F_\alpha^0 \equiv F_\alpha$, in previous definitions. Also, the index "i" in (4.3') is not the same as the index (r) defined earlier, referring to different branches of a single pre-image of the set $\{\mathcal{L}_\alpha^{(r)}\} \subset \mathcal{L}_\alpha \cap (DM)_0$. Thus, a generic central generating function can carry up to two indices $\{i, (r)\}$ to be fully identified.

Finally, since central potentials are defined modulo exact differentials on M, it is important to emphasize that *each nonstandard central equation*, with its appropriate nonstandard consistency conditions, *is defined with respect to a choice of the corresponding nonstandard central potential*. We may fix the standard one by setting $Z_0|_{T^0M} \equiv 0$, but similar choices for the nonstandard ones are not necessarily the best. In Section 6, Proposition 6.1', we describe a consistent choice for all Z_i's which is suitable for the mixed composition of central generating functions.

Such compositions, as we shall see, neatly exhibit the full symplectic and affine geometry of M itself, a fact having some interesting bearings on problems in quantization and semiclassics.

Examples: Standard cases only. We show the explicit map $z[f_\alpha] \equiv F_\alpha : M \to (TM)_0$, given by each generating function f_α, and the canonical relation $(m_-, m_+) \in DM$. To get rid of factors of 2, we often rescale and identify $f_\alpha \equiv 2f$. On $\mathbb{R}^2$, using (3.4) and (4.3), F is written as

$$v_p = \frac{\partial f}{\partial q}, \qquad v_q = -\frac{\partial f}{\partial p} \quad \text{or} \quad \vec{\xi}_\alpha = 2\vec{v} = -J\left[\frac{\partial f_\alpha}{\partial \vec{x}}\right], \tag{4.4}$$

where J is the symplectic matrix on $\mathbb{R}^2$. Composing with (3.3), we have the canonical relation:

$$\vec{x}_\pm = \vec{x} \mp J\left[\frac{\partial f}{\partial \vec{x}}\right] = \vec{x} \mp \frac{1}{2}J\left[\frac{\partial f_\alpha}{\partial \vec{x}}\right], \tag{4.5}$$

see [25,27]. On the torus, (4.4) and (4.5) are valid, but we impose a *standard consistency condition* which amounts to constraining the map (4.4) onto $(T\mathcal{T}^2)_0$ only:

$$\left|\frac{\partial f}{\partial q}\right|, \left|\frac{\partial f}{\partial p}\right| < \frac{\pi}{2} \quad \text{on } U \subset \mathcal{T}^2. \tag{4.6}$$

On S^2, by (3.6) and (4.3), $F : S^2 \to (TS^2)_0$ is written in polar coordinates as:

$$v = S^{-1}(S^0(f)), \qquad \beta = -T^{-1}\left\{\frac{S_\theta(\partial f/\partial\theta)}{\partial f/\partial\varphi}\right\} + \left(1 - \text{Sign}\left(\frac{\partial f}{\partial\varphi}\right)\right)\frac{\pi}{2}, \tag{4.7}$$

where, using the contravariant metric on S^2, we define the symbols

$$S^0(f) := \|\mathrm{d}f\| \equiv \sqrt{\left(\frac{\partial f}{\partial\theta}\right)^2 + \left(\left(\frac{1}{S_\theta}\right)\left(\frac{\partial f}{\partial\varphi}\right)\right)^2}, \qquad C^0(f) := \sqrt{1 - (S^0(f))^2}.$$

Notice that (4.7) has real solutions only if f satisfies the *consistency condition*

$$S^0(f) \equiv \|\mathrm{d}f\| < 1 \quad \text{on } U \subset S^2. \tag{4.8}$$

Composing with the symmetric exponential map (3.5), we obtain

$$\theta_\pm = C^{-1}\left(C_\theta C^0(f) \mp \frac{\partial f}{\partial\varphi}\right),$$
$$\varphi_\pm = \varphi \mp T^{-1}\left\{\frac{S_\theta(\partial f/\partial\theta)}{\alpha_\pm}\right\} + (1 - \text{Sign}(\alpha_\pm))\frac{\pi}{2}, \tag{4.9}$$

where $\alpha_\pm = S_\theta^2 C^0(f) \pm C_\theta(\partial f/\partial\varphi)$, as the local expression for the canonical relation (m_-, m_+) which is generated by $f_\alpha \equiv 2f \in \mathcal{C}^k_{\mathbb{R}}(S^2)$, satisfying condition (4.8).

On H^2, by (3.6) and (3.7) and the central equation, the map $F : H^2 \to TH^2$ is written as

$$\mu = \tilde{S}^{-1}(\tilde{S}^0(f)), \qquad \beta = -T^{-1}\left\{\frac{\tilde{S}_\rho(\partial f/\partial\rho)}{\partial f/\partial\varphi}\right\} + \left(1 - \text{Sign}\left(\frac{\partial f}{\partial\varphi}\right)\right)\frac{\pi}{2} \tag{4.10}$$

in polar coordinates, and this requires no consistency condition. Once more, we have defined

$$\tilde{S}^0(f) := \|\mathrm{d}f\| \equiv \sqrt{\left(\frac{\partial f}{\partial\rho}\right)^2 + \left(\left(\frac{1}{\tilde{S}_\rho}\right)\left(\frac{\partial f}{\partial\varphi}\right)\right)^2}, \qquad \tilde{C}^0(f) := \sqrt{1 + (\tilde{S}^0(f))^2}.$$

Finally, composing with the symmetric exponential map we have the local expression

$$\rho_\pm = \tilde{C}^{-1}\left(\tilde{C}_\rho\tilde{C}^0(f) \pm \frac{\partial f}{\partial\varphi}\right), \quad \varphi_\pm = \varphi \mp T^{-1}\left\{\frac{\tilde{S}_\rho(\partial f/\partial\rho)}{\gamma_\pm}\right\} + (1 - \text{Sign}(\gamma_\pm))\frac{\pi}{2}, \tag{4.11}$$

where $\gamma_\pm = \tilde{S}_\rho^2\tilde{C}^0(f) \pm \tilde{C}_\rho(\partial f/\partial\varphi)$, for the corresponding canonical relation on H^2.

5. Central actions and relations

We saw in the last paragraph that a function $f^i_\alpha \in \mathcal{C}^k_{\mathbb{R}}(M)$ can locally be taken as central generating function of a canonical relation $\Lambda^i_\alpha \subset (\widetilde{TM})_i$, provided (4.2') holds. This is a "graphical" condition over $U \subset M \simeq \widetilde{T^i M} \subset (\widetilde{TM})_i$ and is therefore written with respect to Λ^i_α. But generically, it is precisely this submanifold that needs to be found given f^i_α. Furthermore, it is important to distinguish which of the various lagrangian submanifolds in $(\widetilde{TM})_i$ do correspond to canonical transformations on M, i.e. which are pre-images, under the symmetric exponential map, of graphs over $M_- \subset DM$.

Again, since we usually start with the central generating function, from a practical point of view we need such a distinction, as well as an alternative to condition (4.2'), written directly in terms of f^i_α. We now proceed in this direction, in the *standard* case. First we obtain the following lemma.

Lemma 5.1. *A function $f \equiv (1/2)f_\alpha \in \mathcal{C}^k_{\mathbb{R}}(M)$, $k \geq 2$, can locally be the standard central generating function of a canonical relation, via central equation, only if it satisfies all consistency conditions required for the definition of the map $F : M \supset U \to (TU)_0$, $\vec{F} : m \mapsto \vec{v} \in (T_m M)_0$, and*

$$\left|\det\left[\frac{\partial F^i}{\partial m^j}\right]\right| < \infty, \quad \forall m \in U \subset M \tag{5.1}$$

for any choice of local coordinates $\{m^i\}$ on U, $\{v^i\}$ on $(T_m M)_0$, with $\{F^i(m) = v^i(\tau)\}$.

Proof. Let $\tau_\alpha \in \Lambda_\alpha \subset (TM)_0$, $P_0(\tau_\alpha) = m_\alpha \in M$, and take local coordinates $\{x^1, \dots, x^{2n}\}$ on a neighborhood $X \subset \Lambda_\alpha$ of the point τ_α, $\{m^1, \dots, m^{2n}\}$ on a neighborhood $U \subset M$ of the point m_α and $\{v^1, \dots, v^{2n}\}$ on a neighborhood $V \subset (T_{m_\alpha} M)_0$ of the point $\vec{v}_\alpha = P_v(\tau_\alpha)$, $P_v : N \to V$, $N = (U \times V) \subset (TM)_0$. Then, any point $\tau \in X \subset N$ is locally written as $\tau \equiv \{x^i(\tau)\} \equiv \{x^i\}$, $\tau \equiv \{m^i(P_0(\tau)), v^i(P_v(\tau))\}$, or $\tau \equiv \{m^i(P_0(\{x^j\})), v^i(P_v(\{x^j\}))\}$.

Now, the graphical condition (4.2) is simply $[\mathrm{d}m^i] = A_0[\mathrm{d}x^j]$, $[A_0^{ij}] = [\partial m^i/\partial x^j]$, satisfying $0 < |\det(A_0)| < \infty$, on τ, that is, $[\mathrm{d}x^i] = A_0^{-1}[\mathrm{d}m^j]$, $0 < |\det(A_0^{-1})| < \infty$, and the failure of (4.2) is written as $|\det(A_0)| = 0 \Leftrightarrow |\det(A_0^{-1})| = \infty$. But since no graphical condition over V is assumed, we have only $[\mathrm{d}v^i] = A_v[\mathrm{d}x^i]$, $[A_v^{ij}] = [\partial v^i/\partial x^j]$, satisfying $|\det(A_v)| < \infty$, on τ, and the same cannot be said of A_v^{-1} because $|\det(A_v)| = 0$ is a real possibility. Thus, $|\det(A_v \cdot A_0^{-1})| \equiv |\det[\partial v^i/\partial m^j]| < \infty$ is a necessary condition for the existence of central generating functions.

If such exist, then they satisfy (4.3) and we can write the map F generated by $f \in \mathcal{C}^k_{\mathbb{R}}(U)$ as $\{v^i(\tau) = F^i(m)\}$, provided f satisfies any required central consistency condition. □

Examples: On $\mathbb{R}^2$, (5.1) becomes $|\det[\partial^2 f]| \equiv |\det[\partial^2 f/\partial x^i \partial x^j]| < \infty$.

On S^2 we get the condition $|\det[\partial^2 f] + (C_\theta/S_\theta)(\partial f/\partial\varphi)(\partial^2 f/\partial\theta\partial\varphi)| < \infty$, on $U \subset S^2$, where $\det[\partial^2 f] = (\partial^2 f/\partial\theta^2)(\partial^2 f/\partial\varphi^2) - (\partial^2 f/\partial\theta\partial\varphi)^2$, provided $(\theta = 0, \pi) \notin U$, and f satisfies $0 < \|\mathrm{d}f\| < 1$. The restriction on θ is easily removable by choosing a new origin for the polar coordinates on S^2, but a new local analysis is needed when $\|\mathrm{d}f\|(m_0) = 0$.

Since $\|\mathrm{d}f\| \to 0$ implies $|\vec{v}| \to 0$, we know these points correspond to fixed points of any canonical transformation which can be generated by f. Thus, a (local) alternative is to expand f around m_0 in linear coordinates and apply the previous flat equation, on a small neighborhood of m_0. Otherwise, we can use that $S_v \simeq v$ to get $|\det[\partial^2 f] + (C_\theta/S_\theta)\{(\partial f/\partial\varphi)(\partial^2 f/\partial\theta\partial\varphi) - (\partial f/\partial\theta)(\partial^2 f/\partial\varphi^2)\}| < \infty$, as the local form of (5.1) on a smaller neighborhood U' of a point $m_0 \in U$ for which $\|\mathrm{d}f\|(m_0) = 0$.

Similarly, on $U \subset H^2$, $(\rho = 0) \notin U$, we write (5.1) locally as $|\det[\partial^2 f] + (\tilde{C}_\rho/\tilde{S}_\rho)(\partial f/\partial\varphi)(\partial^2 f/\partial\rho\partial\varphi)|(\tilde{S}_\rho \tilde{S}^0(f)\tilde{C}^0(f))^{-1} < \infty$, if $\|\mathrm{d}f\| > 0$, and $|\det[\partial^2 f] + (\tilde{C}_\rho/\tilde{S}_\rho)\{(\partial f/\partial\varphi)(\partial^2 f/\partial\rho\partial\varphi) - (\partial f/\partial\rho)(\partial^2 f/\partial\varphi^2)\}|(\tilde{S}_\rho)^{-2} < \infty$ on a smaller neighborhood of the point m_0 for which $\|\mathrm{d}f\|(m_0) = 0$. Or we can expand f around m_0 in linear coordinates and use the flat equation.

Lemma 5.1 provides a necessary, but not sufficient condition for the existence of central generating functions, generically. In other words, when $\Lambda \subset (TM)_0$ is a generic canonical relation, it is possible that $|\det(A_0)| = |\det(A_v)| = 0$, on $\tau \in X \subset \Lambda$. That is, we do not necessarily have a graph over either M or $T_m M$, on $\tau = (m, \vec{v}) \in \Lambda$, generically. Even when restricting to canonical transformations on M, it is possible that $\det(A_0) = \det(A_v) = 0$, if $\dim(M) \geq 4$. To see this, consider the simple example: $M = \mathbb{R}^2 \times \mathbb{R}^2$, $\omega = \omega_{(1)} \oplus \omega_{(2)}$, $\alpha = \mathcal{R}_0^{(1)} \otimes \mathrm{id}^{(2)}$. Furthermore, (5.1) does not distinguish between canonical relations and transformations. In this respect, we extend Lemma 5.1 to the following lemma.

Lemma 5.2. *A function $f \equiv (1/2) f_\alpha \in \mathcal{C}^k_{\mathbb{R}}(M)$, $k \geq 2$, can locally be the standard central generating function of a canonical transformation on M only if it satisfies condition (5.1), besides any central consistency condition required for the definition of the map $F : M \supset U \to (TU)_0$, $m \mapsto \vec{F}(m) \in (T_m M)_0$, via the central equation, and*

$$0 < \left| \det \left[\frac{\partial(\exp_m(-\vec{F}(m))^i)}{\partial m^j} \right] \right| < \infty, \quad \forall m \in U \subset M \tag{5.2}$$

for any choice of local coordinates on M.

Proof. Consider $\Lambda_\alpha = \Phi_0^{-1}(\mathcal{L}_\alpha)$, where $\mathcal{L}_\alpha$ is a lagrangian graph over $M_- \subset DM$. Let $\gamma = \Phi_0(\tau) \in Y \subset L_\alpha$, Y a neighborhood of $\gamma_\alpha = \Phi_0(\tau_\alpha)$ and consider local coordinates $\{y^1, \ldots, y^{2n}\}$ on Y. Similarly, take $\{m_-^1, \ldots, m_-^{2n}\}$ as local coordinates on a neighborhood $U_- \subset M_-$ of the point $m_-^\alpha = P_-(\tau_\alpha) = P_-(\gamma_\alpha)$.

Then, $\mathcal{L}_\alpha$ is locally a graph over M_- iff $[\mathrm{d}m_-^i] = B_-[\mathrm{d}y^i]$, where $[B_-^{ij}] = [\partial m_-^i/\partial y^j]$ satisfies $0 < |\det(B_-)| < \infty$. Otherwise, if $\mathcal{L}_\alpha$ is not a graph over M_-, from the definition of B_-, $\det(B_-) = 0$.

Since Φ_0 is a diffeomorphism, we can rewrite above condition as $0 < |\det(A_-)| < \infty$, where $[A_-^{ij}] = [\partial m_-^i/\partial x^i]$, since $A_- = B_- \cdot d\Phi_0$. On the other hand, if Λ_α is a graph over $M \simeq T^0 M$, this can be further rewritten as $0 < |\det(A_-^0)| < \infty$, where $[(A_-^0)^{ij}] = [\partial m_-^i/\partial m^j]$, since $A_-^0 = A_- \cdot A_0^{-1}$ and $|\det(A_0^{-1})| < \infty$.

But since $m_- = P_-(\tau) = \exp_m(-\vec{v})$, we get (5.2), provided Λ_α satisfies (4.2) and τ is given by the central map $\vec{F}(m) = \vec{v}$, generated by f via the central equation. □

Example: On $\mathbb{R}^2$, from (3.3) and (4.4), (5.2) becomes $0 < |1 + \det[\partial^2 f]| < \infty$, but since (5.1) must already be satisfied, we single out the new condition as $\det[\partial^2 f] \neq -1$, $\forall \vec{x} \in U \subset \mathbb{R}^2$. However, already on S^2 or H^2, the explicit form of (5.2) in local coordinates, for generic f, becomes rather long and it is much simpler to check it directly, for each specific f, using the specific expressions for $m_-(m)$ obtained from (4.9) or (4.11).

Conditions similar to (5.1) and (5.2) apply to nonstandard functions, as well. It is important to emphasize, once again, that any function satisfying the conditions of Lemma 5.2 does not necessarily generate a canonical transformation on M, since these conditions do not comprise a sufficient set, generically. A more complete analysis is needed for such characterization, which lies outside the scope of this work. Ultimately, though, we can check explicitly the consistency of the implicit map $m_+(m_-)$ obtained from any function f on $U \subset M$, via the central equation and the symmetric exponential map. These functions shall be singled out from the following definition.

Definition 5.1. A function $f \equiv (1/2) f_\alpha \in \mathcal{C}^k_{\mathbb{R}}(M)$, $k \geq 2$, which is (locally, on $U \subset M$) the central generating function of a canonical transformation on M is henceforth referred to simply as a (local) *central action* on M. The set of all such functions is denoted $\mathcal{A}_c(U)$. Specifically, $\mathcal{A}^0_c(U)$ for standard actions. More generally, a function which locally generates a canonical relation on M, via the central equation, shall be called a (local) *central relation* on M, whose set is denoted $\mathrm{Rel}_c(U)$.

Of course, $\mathcal{A}_c(U_1) \subset \mathcal{A}_c(U_2)$ if $U_1 \supset U_2$. Thus, $\forall U \subset M$, $\mathcal{A}_c(U) \supset \mathcal{A}_c(M)$ the space of central actions on M, clearly nonempty since every function $f_\alpha = -th$, $h \in \mathcal{C}^k_{\mathbb{R}}(M)$, is a central action, for t sufficiently small. On the other hand, since central relations must satisfy (5.1), but not necessarily (5.2), $\mathcal{A}_c(U) \subset \mathrm{Rel}_c(U)$.

More generally, one might still wish to be free from constraints imposed by central caustics. For functions on M, this is generically impossible.

On the other hand, inspired by some works in symplectic topology [6,7,39], we could allow the generating functions to be defined on $M \times \mathbb{R}^d$, instead, using the extra variables to analyse the behaviour at those critical points. But the study of their general definitions and properties is not to be found here. In an independent context, a very important and particular case of such "extended" functions shall be seen in Section 8 and 9, where $d = 1$.

Back to functions on M, our main interest, in what follows, is concerned with their compositions. For central actions we need only worry about the presence of caustics, but *for canonical relations we must also worry about the possibility of their compositions being well defined* (clean products) [10,14,42,43]. Thus, for simplicity, we shall not pursue on this broader context here, focusing instead on the compositions of central actions properly, from now on.

But we emphasize that, whenever well defined, the following rules of composition apply for central relations as well.

Illustrations: The simplest examples of central actions are those which generate uniform translations on flat space. Let $\alpha \equiv T_{\vec{\xi}} : \mathbb{R}^2 \to \mathbb{R}^2$, by $\vec{x} \mapsto \vec{x} + \vec{\xi}$, which corresponds to the lagrangian plane $\vec{v} = \vec{\xi}/2$, a constant, in $T\mathbb{R}^2$. From (4.5) one has $\vec{\xi} = -J(\partial f_\alpha / \partial \vec{x})$, integrated as $f_\alpha(\vec{x}) = (J \cdot \vec{\xi}) \cdot \vec{x}$ modulo constants, rewritten as a skew-product $f_\alpha(\vec{x}) = \vec{\xi} \wedge \vec{x}$.

In particular (when $\vec{\xi} = 0$) the null action (or any constant) generates the identity. And this is true for every symmetric symplectic space M.

The next simplest examples, still on flat spaces, are the homogeneous quadratic functions $f_\alpha(\vec{x}) = \vec{x}^{\mathrm{T}} B\vec{x}$, where B is a symmetric matrix, or equivalently $f_\alpha(\vec{x}) = \beta p^2 + \beta' q^2 + 2\gamma pq$, where $\beta, \beta', \gamma \in \mathbb{R}$. When $\gamma = 0$, $\beta = \beta' = -\tan(\lambda/2)$, f_α generates a rotation by an angle λ through the origin. Notice that f_α, as well as (5.1), diverges everywhere when $\lambda = \pm\pi$. This is a reflection at the origin, $\mathcal{R}_0$, and the corresponding lagrangian submanifold in $T\mathbb{R}^2$ is $\Lambda_{\mathcal{R}_0} \equiv T_0\mathbb{R}^2$ which is not a graph over $T^0\mathbb{R}^2$ anywhere. When $\beta = \beta' = 0$, $\gamma = -\tanh(\lambda/2)$, f_α generates a pure hyperbolic transformation with stable/unstable submanifold coinciding with the p/q axis. Now (5.1) is always satisfied, but (5.2) fails asymptotically as $|\lambda| \to \infty$. It is not hard to see that every quadratic central action generates an element of the homogeneous symplectic group on $\mathbb{R}^2$ and, conversely, every such element which can be centrally generated, is done by a quadratic central action (Cayley transform). This is not a property of the more familiar generating functions of mechanics on $\mathbb{R}^2$, e.g. $f(q_-, q_+)$, for which generic translations on M are also generated by quadratic functions. Adding the former two examples, $f_\alpha(\vec{x}) = \vec{\xi} \wedge \vec{x} + \vec{x}^{\mathrm{T}} B\vec{x}$ generate elements of the inhomogeneous symplectic group.

On S^2, the simplest examples are the central actions for rotations. Here, standard case only. Let $\alpha \equiv \alpha(p, 2\gamma)$ be an element of the group of rotations SO(3) acting on S^2, whose pole (fixed point) is p and whose angle of rotation is 2γ. Taking local polar coordinates for $m \equiv (\theta, \varphi)$ and $p \equiv (\chi, \varepsilon)$, and $\gamma \in [-\pi/2, \pi/2]$, the central action is written as

$$f_\alpha(m) = -2S^{-1}\{S_\gamma[C_\chi C_\theta + S_\chi S_\theta C_{(\varphi-\varepsilon)}]\}. \tag{5.3}$$

Although it looks complicated in local coordinates, $f \equiv (1/2) f_\alpha$ has the simple geometrical interpretation shown in Fig. 1(a). If we notice that $C_\chi C_\theta + S_\chi S_\theta C_{(\varphi-\varepsilon)} = C_y$,

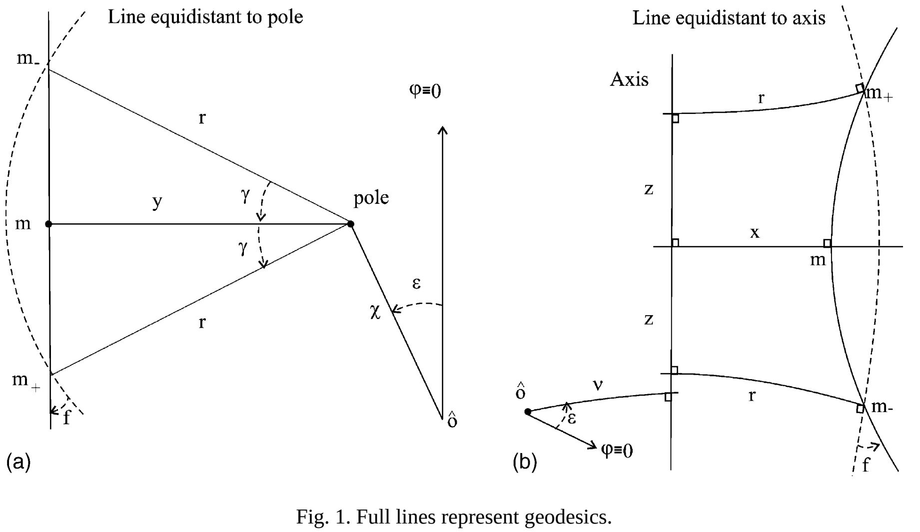

Fig. 1. Full lines represent geodesics.

$y = \text{distance}(m, p)$, the map $F_\alpha : S^2 \to (TS^2)_0$ is given by (4.7) as

$$v = C^{-1}\left\{\frac{C_\gamma}{\sqrt{1-(S_\gamma C_y)^2}}\right\} \equiv C^{-1}\left\{\frac{C_\gamma}{C_f}\right\},$$
$$\beta = T^{-1}\left\{\frac{S_\chi C_\theta C_{(\varphi-\varepsilon)} - C_\chi S_\theta}{S_\chi S_{(\varphi-\varepsilon)}}\right\} + (1 - \text{Sign}(\varphi - \varepsilon))\frac{\pi}{2} \tag{5.4}$$

and the implicit transformation $m_+(m_-)$ generated by f_α is given by (4.9) as

$$\theta_\pm = C^{-1}\left\{\frac{C_\gamma C_\theta \pm S_\gamma S_\theta S_\chi S_{(\varphi-\varepsilon)}}{\sqrt{1-(S_\gamma C_y)^2}}\right\},$$
$$\varphi_\pm = \varphi \pm T^{-1}\left\{\frac{S_\gamma[C_\chi S_\theta - S_\chi C_\theta C_{(\varphi-\varepsilon)}]}{\lambda_\pm}\right\} + (1 - \text{Sign}(\lambda_\pm))\frac{\pi}{2}, \tag{5.5}$$

where $\lambda_\pm = C_\gamma S_\theta \pm S_\gamma C_\theta S_\chi S_{(\varphi-\varepsilon)}$. Notice that when $m \to p$ or $\bar{p}$, $v \to 0$ and m is a fixed point of the transformation (5.5), as expected, $\forall\gamma \in [0, \pi/2)$. However, when $\gamma = \pi/2$ the transformation (5.5) is not well defined. Again, in this case $\alpha(p, 2\gamma) = \alpha(p, \pi) = \mathcal{R}_p : S^2 \to S^2$, and the corresponding lagrangian submanifold in $(TS^2)_0$ is $\Lambda_{\mathcal{R}_p} \equiv (T_pS^2)_0$, which is not a graph over T^0S^2 anywhere. We can also see this singularity using (5.1), when $\gamma = \pi/2$. To see that $m \to p$ is not a caustic singularity, for $\gamma \neq \pi/2$, one can check directly in (5.1). Similarly, expanding f around m in linear coordinates (p, q), gives $f_\alpha(m) = \text{constant} + (T_\gamma)y^2 + \text{o}(y^4) \simeq (T_\gamma)y^2$, with $y^2 = p^2 + q^2$, which is the central action for rotations on the plane.

On H^2, we now consider the central actions for some elements $\alpha \in \text{SO}(2, 1)$. A subclass of such elements is characterized by those $\alpha = \alpha(p, 2\gamma)$, which are real rotations on H^2 around the real pole $p \in H^2$, through an angle 2γ, $\gamma \in [-\pi/2, \pi/2]$. The corresponding central actions and canonical transformations are analogous to the spherical ones.

However, in opposition to the real hyperbolic rotations, stand the ideal hyperbolic rotations. These can be characterized in two ways. We may consider $\alpha = \alpha(\tilde{p}, 2\gamma)$, $\gamma \in [-\pi/2, \pi/2]$, $\tilde{p}$ an ideal fixed point (a point in $\mathbb{R}^3$ satisfying $x^2 + y^2 - z^2 = 1$, while a real point in $H^2 \subset \mathbb{R}^3$ satisfies $z^2 - (x^2 + y^2) = 1$). Alternatively, we take $\alpha = \tilde{\alpha}(\nu, \varepsilon, z)$ where $\nu \in [0, \infty)$, $z \in (-\infty, \infty)$, $\varepsilon \in [0, 2\pi]$; (ν, ε) being the coordinates of a real axis in H^2, i.e. a geodesic in H^2 s.t. ν is its distance to the origin and ε is the angle this minimal geodesic arc makes at the origin, as shown in Fig. 1(b). We interpret $|z| < \infty$ as a geodesic segment along this axis, its sign determining a particular orientation for it [35]. The corresponding central action is given by

$$f_{\tilde{\alpha}}(m) = 2S^{-1}\{\tilde{S}_z[\tilde{C}_\nu \tilde{S}_\rho C_{(\varphi-\varepsilon)} - \tilde{S}_\nu \tilde{C}_\rho]\}, \tag{5.6}$$

where $\tilde{C}_\nu \tilde{S}_\rho C_{(\varphi-\varepsilon)} - \tilde{S}_\nu \tilde{C}_\rho = \tilde{S}_x$, $x = \text{distance}(m, \text{axis})$. From (5.6) $f_{\tilde{\alpha}}$ exists only when

$$|\tilde{S}_z \tilde{S}_x| \le 1 \Leftrightarrow 0 \le x \le \tilde{S}^{-1}\left(\frac{1}{|\tilde{S}_z|}\right), \tag{5.7}$$

but in this neighborhood of the axis, it is a well-defined real function whose simple geometric interpretation is shown in Fig. 1(b). From (4.10), the map $F_{\tilde{\alpha}} : H^2 \to TH^2$ is given by

$$\mu = \tilde{C}^{-1}\left\{\frac{\tilde{C}_z}{\sqrt{1-(\tilde{S}_z\tilde{S}_x)^2}}\right\} \equiv \tilde{C}^{-1}\left\{\frac{\tilde{C}_z}{C_f}\right\},$$
$$\beta = T^{-1}\left\{\frac{\tilde{C}_\nu\tilde{C}_\rho C_{(\varphi-\varepsilon)} - \tilde{S}_\nu\tilde{S}_\rho}{\tilde{C}_\nu S_{(\varphi-\varepsilon)}}\right\} + (1+\mathrm{Sign}(\varphi-\varepsilon))\frac{\pi}{2} \tag{5.8}$$

and the $f_{\tilde{\alpha}}$-generated intrinsic transformation $m_+(m_-)$ is given by (4.11) as

$$\rho_\pm = \tilde{C}^{-1}\left\{\frac{\tilde{C}_z\tilde{C}_\rho \pm \tilde{S}_z\tilde{S}_\rho\tilde{C}_\nu S_{(\varphi-\varepsilon)}}{\sqrt{1-(\tilde{S}_z\tilde{S}_x)^2}}\right\},$$
$$\varphi_\pm = \varphi \pm T^{-1}\left\{\frac{\tilde{S}_z[\tilde{C}_\nu\tilde{C}_\rho C_{(\varphi-\varepsilon)} - \tilde{S}_\nu\tilde{S}_\rho]}{\xi_\pm}\right\} + (1+\mathrm{Sign}(\xi_\pm))\frac{\pi}{2}, \tag{5.9}$$

where $\xi_\pm = \tilde{C}_z\tilde{S}_\rho \pm \tilde{S}_z\tilde{C}_\rho\tilde{C}_\nu S_{(\varphi-\varepsilon)}$. The transformation (5.9) describes finite motions along lines equidistant to the axis (ν, ε), as shown in Fig. 1(b). We see that (5.9) has no real fixed point and is also free of central caustics, i.e. it is well defined everywhere on the same neighborhood of the axis (ν, ε) on which $f_{\tilde{\alpha}}$ is well defined. In other words, the lagrangian submanifold $\Lambda_{\tilde{\alpha}} \subset TH^2$ is a graph over this neighborhood; one can check explicitly that (5.1) does not diverge anywhere. Notice that, as $x \to 0$, m, m_+ and m_- all lie on the axis, with $z = \mathrm{distance}(m, m_+)$. However, as $|z| \to \infty$, (5.2) fails asymptotically, in agreement with (5.7), which implies $x = 0$, when $|z| = \infty$.

6. Composition of central actions

Having explored the symplectic structure on central groupoids, we now add their algebraic structure in order to answer the following question: Let $f_{\alpha_1}, f_{\alpha_2} \in \mathcal{C}^k_{\mathbb{R}}(M)$ locally be the central actions for two canonical transformations, respectively, $\alpha_1, \alpha_2 : M \to M, m \mapsto \alpha_i(m)$. Since $\alpha = \alpha_2(\alpha_1)$ is another canonical transformations on M, what is, locally, its central action? In other words, How do central actions compose? We start by considering the following definition.

Definition 6.1. Let ζ be a (local) symplectic potential for ω on $U_\zeta \subset M$. Then, on $DU_\zeta \subset DM$, $\delta\zeta := P_+^*(\zeta) - P_-^*(\zeta)$ is a local symplectic potential for $\delta\omega$ and $Z_\zeta := \Phi_0^*(\delta\zeta)$ is locally a symplectic potential for Ω_0 on $W_\zeta \subset (TM)_0$, $\Phi_0(W_\zeta) \subset DU_\zeta$, to be called an *additive potential*.

The reason for this name is obvious once we realize that

$$Z_\zeta(\tau_1 \odot \tau_2) \approx Z_\zeta(\tau_1) + Z_\zeta(\tau_2). \tag{6.1}$$

Here, as in Definition 3.2, this is a shorthand notation for the fact that the local 1-form $Z_\zeta \oplus Z_\zeta \ominus Z_\zeta$ vanishes on the graph of the groupoid composition.

Such additive potentials do not coincide with the central potentials, i.e. Z_ζ does not satisfy property (4.1), even locally, and we cannot use it to define central generating functions directly. On the other hand, the central potentials are not additive, as in (6.1), but in order to compose central actions, we now realize that, on W_ζ,

$$Z_\zeta = Z_0 + \mathrm{d}Q_\zeta, \quad Q_\zeta \equiv Q^0_\zeta \in \mathcal{C}^k_{\mathbb{R}}(W_\zeta). \tag{6.2}$$

Therefore, if $\tau_i \in \Lambda_{\alpha_i}$, with $P_0(\tau_i) = m_i \in U$, is locally generated by the standard central action $f_{\alpha_i} \in \mathcal{C}^k_{\mathbb{R}}(M)$, then on $(TU)_0 \cap W_\zeta : Z_\zeta(\tau_i) = \mathrm{d}f_{\alpha_i}(m_i) + \mathrm{d}Q_\zeta(\tau_i)$, combining (4.3) and (6.2). Hence, if Λ_α satisfies (4.2) on $(TU)_0$, for $\alpha = \alpha_2(\alpha_1)$, and $\tau = \tau_1 \odot \tau_2 \in \Lambda_\alpha$, with $P_0(\tau) = m \in U$, is locally generated by the standard central action $f_\alpha \in \mathcal{C}^k_{\mathbb{R}}(M)$, then, on $W_\zeta \cap (TU)_0$, from (6.1),

$$\mathrm{d}f_\alpha(m) \approx \mathrm{d}f_{\alpha_1}(m_1) + \mathrm{d}f_{\alpha_2}(m_2) + \mathrm{d}\chi_\zeta(\tau_1, \tau_2), \tag{6.3}$$

where we define the standard phase function $\chi_\zeta \equiv \chi^0_\zeta : (TU)^2_0 \to \mathbb{R}$, locally by

$$\chi_\zeta(\tau_1, \tau_2) = Q_\zeta(\tau_1) + Q_\zeta(\tau_2) - Q_\zeta(\tau_1 \odot \tau_2). \tag{6.4}$$

Now we apply the crucial result stated in the following proposition.

Proposition 6.1. *The function χ_ζ defined above (6.4) is well defined on the whole $(TM)^2_0$ and independs on choices of local symplectic potentials on M. Actually it coincides, modulo constants, with the symplectic area of a standard geodesic triangle on M, i.e. which can be defined by elements in $(TM)^2_0$, determined by its midpoints m, m_1 and m_2. This area shall be denoted by $\Delta_0(m, m_1, m_2)$.*

Proof. Here we rely on a mathematical construction [1,3] which consists of building an S^1-principal fiber bundle over a symplectic manifold (M, ω) with connection α whose curvature is ω/λ. We denote this as $S^1 \to (SM, \alpha) \xrightarrow{\pi} (M, \omega/\lambda)$, $\mathrm{d}\alpha = \pi^*(\omega/\lambda)$. Such bundle is well defined only if $(1/2\pi\lambda)\oint_{\mathcal{B}} \omega = p \in \mathbb{Z}$, where $\mathcal{B}$ is any oriented, closed two-surface without boundary on M. Here, λ is an auxiliary constant which can be set to zero in the end.

To extend this construction to the double phase space [48], we identify $SDM \equiv (\overline{SM}_- \times \overline{SM}_+)/S^1$, $(\overline{SM}_\pm, \mp\alpha)$ being the same principal bundles over M, but for opposite connections, and the quotient is taken with respect to the diagonal action of $S^1 \subset \mathcal{T}^2$ on $\overline{SM}_- \times \overline{SM}_+$. Actually, we have chosen the connection whose curvature is $-\delta\omega/\lambda$, so we should perhaps denote this bundle by $\overline{SDM}$, but to simplify the notation we keep to SDM. Thus, $S^1 \to (SDM, [-\delta\alpha]) \xrightarrow{[\pi]} (DM, -\delta\omega/\lambda)$, $\mathrm{d}[-\delta\alpha] = [\pi]^*(-\delta\omega/\lambda)$, whose elements are denoted by $[\sigma, \sigma']$, where $(\sigma, \sigma') \in \overline{SM}_- \times \overline{SM}_+$.

Choosing identity elements of the form $[\sigma, \sigma]$, we extend [51] the groupoid composition from DM to SDM by $[\sigma, \sigma'] \odot [\sigma', \sigma''] = [\sigma, \sigma'']$. Pulling back SDM by the (restricted) symmetric exponential map, we get a (trivial) bundle over $(TM)_0$, denoted $(STM)_0 \equiv \Phi^*_0(SDM) \xrightarrow{\pi_0 \times \rho_0} (TM)_0 \times SDM$.

Next, consider sections $\varepsilon_0 : (TM)_0 \to (STM)_0$ which are obtained via the parallel transport along the P_0-fibers of identity elements in $(STM)_0$, the latter being fixed by the above choice on *SDM*. These polarized sections are such that $\rho_0(\varepsilon_0(\tau)) = [\sigma', \sigma'']$, where (σ', σ'') are the endpoints of a horizontal lift in *SM* of the short geodesic from m' to m'', centered on $m = P_0(\tau)$, for $\Phi_0(\tau) = (m', m'')$.

Hence, if $(\tau_1, \tau_2) \in (TM)_0^2$, then $\varepsilon_0(\tau_1) \odot \varepsilon_0(\tau_2) = \varepsilon_0(\tau_1 \odot \tau_2)K_0(\tau_1, \tau_2)$, where $K_0 : (TM)_0^2 \to S^1$ is the holonomy in *SM* over the triangle of composition on M. That is, $K_0(\tau_1, \tau_2) = \exp\{(\sqrt{-1}/\lambda)A_0(\tau_1, \tau_2)\}$, where the standard triangular area function A_0 is well defined and continuous on $(TM)_0^2$. This means that $K_0(\tau_1, \tau_2)$ is well defined on the whole $(TM)_0^2$, as well as being independent of local expressions for the connection $[-\delta\alpha]$ and its pull-back $\bar{\alpha}_0$.

But the pull-back connection $\bar{\alpha}_0$ can locally be written as $\bar{\alpha}_0 \simeq \mathrm{d}\theta - (1/\lambda)Z_\zeta$, where θ is the fiber coordinate, for a local choice of symplectic potential $\omega \simeq \mathrm{d}\zeta$. Over the P_0-fibers, these can locally be rewritten as $\bar{\alpha}_0|_m \simeq (\mathrm{d}\theta - (1/\lambda)(Z_\zeta - Z_0))|_m \simeq (\mathrm{d}\theta - (1/\lambda)\,\mathrm{d}Q_\zeta)|_m$, thus, provided $Q_\zeta|_{T^0M} \equiv 0$ and for a local representation of identity elements as $(m, \vec{0}; 0)$, the trivializing sections ε_0 can locally be written as $\varepsilon_0(\tau) \simeq (\tau; \exp\{(\sqrt{-1}/\lambda)Q_\zeta(\tau)\})$ and therefore the holonomy phase $A_0(\tau_1, \tau_2)$ is locally identified as $\chi_\zeta(\tau_1, \tau_2) = Q_\zeta(\tau_1) + Q_\zeta(\tau_2) - Q_\zeta(\tau_1 \odot \tau_2)$.

On the other hand, denoting a point in $(TM)_0^2$ by $(m_1, v_1; m_2, v_2)$, with $\tau_1 \odot \tau_2 = \tau_3 = (m_3, v_3)$, we can almost everywhere eliminate the three vectors by $\exp_{m_1}(-v_1) = \exp_{m_3}(-v_3) = a$, $\exp_{m_1}(v_1) = \exp_{m_2}(-v_2) = b$, $\exp_{m_2}(v_2) = \exp_{m_3}(v_3) = c$, and $\Phi_0(m_1, v_1) = (a, \mathcal{R}_{m_2}\mathcal{R}_{m_3}(a) = b)$, $\Phi_0(m_2, v_2) = (b, \mathcal{R}_{m_3}\mathcal{R}_{m_1}(b) = c)$, $\Phi_0(m_3, v_3) = (a, \mathcal{R}_{m_2}\mathcal{R}_{m_1}(a) = c)$, the exception being those midpoint triplets $\mu_{123} = (m_1, m_2, m_3)$ for which the fixed point set of $\mathcal{R}_{m_1}\mathcal{R}_{m_2}\mathcal{R}_{m_3}$ has nontrivial dimension. But we can assume, based on the lower dimensional examples, that the set of such singular midpoint triplets has codimension always greater than 1. In other words, the map $\Psi_0 : (TM)_0^2 \to (M \times M \times M)_{\text{midpoints}}$ is invertible on its image set, except for a subset of codimension greater than 1.

Hence, if μ_{123} is a regular midpoint triplet, $\Delta_0(\mu_{123}) = \Delta_0(m_1, m_2, m_3) = A_0(\Psi_0^{-1}(\mu_{123}))$. Otherwise, if μ_{123} is a singular midpoint triplet, consider any continuous family of regular midpoint triplets $\mu'_{123}(\epsilon)$, $\epsilon > 0$ s.t. $\mu'_{123}(0) = \mu_{123}$. These families always exist and, for any $\epsilon \neq 0$, Ψ_0^{-1} is well defined and continuous, so that $\Delta_0(\mu'_{123}(\epsilon))$ is a continuous function of ϵ. It follows that $\Delta_0(\mu_{123}) = \Delta_0(m_1, m_2, m_3) = \lim_{\epsilon \to 0}\{\Delta_0(\mu'_{123}(\epsilon))\}$, for any family $\mu'_{123}(\epsilon) \to \mu_{123}$.

Therefore, the holonomy can be identified with $\exp\{(\sqrt{-1}/\lambda)\Delta_0(m, m_1, m_2)\}$, for $m_i = P_0(\tau_i)$, $m = P_0(\tau_1 \odot \tau_2)$, where $\Delta_0(m, m_1, m_2)$ stands for the symplectic area, modulo $2\pi\lambda$, of a standard geodesic triangle with given midpoints, for which $\chi_\zeta(\tau_1, \tau_2)$ is thus a local expression. □

Let us remark that in the simplest case of $\mathbb{R}^{2n}$, the midpoint triangular area Δ is a unique well-defined function on the whole $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Generically, however, Δ_0 is well defined only on a subset $U \subset M \times M \times M$, as exemplified below for the torus, the sphere and the hyperbolic plane.

In every case, though, for any triplet of points for which Δ_0 is well defined, modulo constants, we get from Propositions 4.1 and 6.1, via (6.3), the main result. We now have the following theorem.

Theorem 6.1. *Let $f_{\alpha_1}, f_{\alpha_2}$ locally be standard central actions for $\alpha_1, \alpha_2 : M \to M$, respectively. If f_α is locally the standard central action for $\alpha = \alpha_2(\alpha_1)$, then*

$$f_\alpha(m) \equiv f_{\alpha_1} \triangle f_{\alpha_2}(m) = \mathrm{Stat}_{(m_1,m_2)}\{f_{\alpha_1}(m_1) + f_{\alpha_2}(m_2) + \Delta_0(m, m_1, m_2)\} \quad (6.5)$$

defining the standard composition of central actions on M, if $\Delta_0(m, m_1, m_2)$ is well defined, up to constants, i.e. if (m, m_1, m_2) stand as midpoints of a standard geodesic triangle.

As a product on $\mathcal{A}^0_{\mathrm{c}}(U)$, $f_{\alpha_1} \triangle f_{\alpha_2}$ may not exist or if so, it may not be unique, for $\Lambda_\alpha \subset (TM)_0$ may not satisfy (4.2) over $U \subset M$, or it may be composed of many sheets in $(TU)_0$.

Furthermore, the composition rule (6.5) is not unique when considering nonstandard central actions as well, if M has closed geodesics. In this case, repeating the steps that led to (6.3), for reducible triples $(i, j; k)$, using (4.3') we get the generalized version:

$$\mathrm{d} f^k_\alpha(m) \approx \mathrm{d} f^i_{\alpha_1}(m_1) + \mathrm{d} f^j_{\alpha_2}(m_2) + \mathrm{d}[\chi_\zeta]^k_{ij}(\tilde\tau_1, \tilde\tau_2), \quad (6.3')$$

where $[\chi_\zeta]^k_{ij}$ is a local representation for a general phase function $(\widetilde{TM})^2_{ij} \to \mathbb{R}$, given by

$$[\chi_\zeta]^k_{ij}(\tilde\tau_1, \tilde\tau_2) = Q^i_\zeta(\tilde\tau_1) + Q^j_\zeta(\tilde\tau_2) - Q^k_\zeta(c^k_{ij}(\tilde\tau_1, \tilde\tau_2)), \quad (6.4')$$

with each Q^i_ζ being a function defined as $\mathrm{d}Q^i_\zeta = Z^i_\zeta - Z_i$, on appropriate subsets $W^i_\zeta \subset (\widetilde{TM})_i$. As in the standard case, we identify this function by the following proposition.

Proposition 6.1'. *For $(i, j; k)$ reducible, and for appropriate choices of Z_i, Z_j, and Z_k, the function $[\chi_\zeta]^k_{ij}$ is well defined on the whole $(\widetilde{TM})^2_{ij}$, independs on local choices of ζ and coincides, modulo constants, with the symplectic area of a reducible geodesic triangles on M which can be defined by elements in $(\widetilde{TM})^2_{ij} \times (\widetilde{TM})_k$ and determined by its midpoints $(m_1, m_2; m)$, in other words, geodesic triangles with given midpoints and sides which are $(i, j; k)$-fold geodesics. Such area shall be denoted by $\Delta^k_{ij}(m; m_1, m_2)$ or generically by $\Delta(m, m_1, m_2)$.*

Proof. This is an elaboration on the proof of Proposition 6.1. Here, we must argue that there is a consistent choice for the Z_j's, such that each $Q^j_\zeta|_{\widetilde{T^jM}} \equiv \gamma^j : M \to \mathbb{R}$ is given by $\exp\{(\sqrt{-1}/\lambda)\gamma^j(m)\} = \eta'_j(m)$, where $\eta'_j(m)$ is the horizontal transport over $M \simeq \widetilde{T^jM}$ of the $L_j(m_0)$ loop holonomy, given a reference point m_0.

Explicitly, if $\exp\{\sqrt{-1}\beta_j\}$ denotes the $L_j(m_0)$ holonomy in SM (its homology class, if irreducible), then $\gamma^j(m) = \int_{\Sigma L'_j(m)} w$, modulo $2\pi\lambda$ and the constant β_j, where $\partial\Sigma L'_j(m) = L'_j(m) = L_j(m_0, m)$ is the reducible loop in M given as $(m_0 \to m) \circ L_j(m) \circ (m \to m_0) \circ L^{-1}_j(m_0)$, with $L_j(m)$ denoting the j-fold geodesic loop based at m.

Each $\tilde\Phi_j : (\widetilde{TM})_j \to DM$ provides a pull-back (trivial) bundle $(S\widetilde{TM})_j \xrightarrow{\rho_j} SDM$. Taking local trivializing sections $\tilde\varepsilon_j : \widetilde{T^jM} \to (S\widetilde{TM})_j$ of the form $\tilde\varepsilon_j = \rho_i^{-1}([\sigma, \sigma'_j])$, where $\sigma'_j = \sigma \cdot \eta'_j(m)$, for $\pi(\sigma) = \pi(\sigma') = m$, locally represented as $\tilde\varepsilon_j(\tilde\tau^j) \simeq (m, [\vec{v}_j]; \exp\{(\sqrt{-1}/\lambda)$ $\gamma^j(m)\})$, then, from a similar analysis to the standard case, the trivializing sections ε_j :

$(\widetilde{TM})_j \to (S\widetilde{TM})_j$ which are obtained from $\tilde{\varepsilon}_j$ via parallel transport along the P_j-fibers are such that $\rho_j(\varepsilon_j(\tilde{\tau})) = [\sigma', \sigma'']$, where (σ', σ'') are the endpoints of a horizontal lift in *SM* of the j-fold geodesic from m' to m'', centered on $m = P_j(\tilde{\tau})$, for $\tilde{\Phi}_j(\tilde{\tau}) = (m', m'')$.

Hence, if $(\tilde{\tau}_1, \tilde{\tau}_2) \in (\widetilde{TM})^2_{ij}$, then $C^k_{ij}(\varepsilon_i(\tilde{\tau}_1), \varepsilon_j(\tilde{\tau}_2)) = \varepsilon_k(c^k_{ij}(\tilde{\tau}_1, \tilde{\tau}_2)) \cdot K^k_{ij}(\tilde{\tau}_1, \tilde{\tau}_2)$, where $C^k_{ij} : (S\widetilde{TM})^2_{ij} \to (S\widetilde{TM})_k$ is the "extension" of the mixed composition c^k_{ij}, defined as $C^k_{ij}(z_1, z_2) := \rho_k^{-1}(\rho_i(z_1) \odot \rho_j(z_2)), z_i \in (S\widetilde{TM})_i$, etc. and $K^k_{ij}(\tilde{\tau}_1, \tilde{\tau}_2) \in S^1$ is the holonomy in *SM* over the $(i, j; k)$-triangle of composition, which, if $(i, j; k)$ is reducible, corresponds to the exponential of its symplectic area.

Furthermore, since any reducible $(i, j; k)$-geodesic triangle stands in a bijective correspondence with a standard one, it follows that the $(i, j; k)$-triangular area is a well-defined function Δ^k_{ij} of the midpoints.

Therefore, the holonomy can be identified as $\exp\{(\sqrt{-1}/\lambda)\Delta^k_{ij}(m; m_1, m_2)\}$, for $m_1 = P_i(\tilde{\tau}_1)$, $m_2 = P_j(\tilde{\tau}_2)$ and $m = P_k(c^k_{ij}(\tilde{\tau}_1, \tilde{\tau}_2))$, where $\Delta^k_{ij}(m; m_1, m_2)$ stands for the symplectic area, modulo $2\pi\lambda$, of the reducible geodesic triangle with given midpoints and sides which are $(k; i, j)$-fold geodesics. This also means that $K^k_{ij}(\tilde{\tau}_1, \tilde{\tau}_2) = \exp\{(\sqrt{-1}/\lambda)\Delta^k_{ij}(m; m_1, m_2)\}$ is well defined on the whole $(\widetilde{TM})^2_{ij}$ and independs on local expressions for the connection $[-\delta\alpha]$ and its pull-backs $\bar{\alpha}_j$.

But, from the form of such pull-back connections and for $Q^j_\zeta|_{\widetilde{T^jM}} \equiv \gamma^j$ and $\varepsilon_j|_{\widetilde{T^jM}} \equiv \tilde{\varepsilon}_j$, as above, it follows that the trivializing sections ε_j can locally be written as $\varepsilon_j(\tilde{\tau}) \simeq (\tilde{\tau}; \exp\{(\sqrt{-1}/\lambda)Q^j_\zeta(\tilde{\tau})\})$ and therefore $[\chi_\zeta]^k_{ij}$ is a local representation for $\Delta^k_{ij}(m; m_1, m_2)$, modulo $2\pi\lambda$. □

The appropriate choices of central potentials referred to in the statement are explained in its proof. Loosely speaking, each Z_j is chosen as the pull-back of a single 1-form Z on *TM*, for each immersion $(\widetilde{TM})_j \to TM$. Using (6.3') we get the following theorem.

Theorem 6.1'. *Let $f^i_{\alpha_1}$, $f^j_{\alpha_2}$ locally be central actions for $\alpha_1, \alpha_2 : M \to M$, respectively. If $(i, j; k)$ is reducible and f^k_α is locally a central action for $\alpha = \alpha_2(\alpha_1)$, then it is given by*

$$f^k_\alpha(m) \equiv f^i_{\alpha_1}\ \Delta^k\ f^j_{\alpha_2} = Stat_{(m_1,m_2)}\{f^i_{\alpha_1}(m_1) + f^j_{\alpha_2}(m_2) + \Delta^k_{ij}(m; m_1, m_2)\} \tag{6.5'}$$

defining the $(i, j; k)$ composition of local central actions on M, provided $(m_1, m_2; m)$ can stand as midpoints for a reducible $(i, j; k)$-geodesic triangle and each action is defined with respect to the choices of central potentials referred to and explained above.

Clearly, (6.5) is a particular case of (6.5'), with $\Delta^0_{00} \equiv \Delta_0$. However, by restricting to the standard composition we have obtained a uniquely defined rule. In extending to generic central actions this rule ceased to be unique, even though each instance is well defined. Thus, for simplicity, when considering multiple compositions, in Section 7, we keep to the standard case.

Finally we remind that, *whenever meaningful* (*clean product* [43]), (6.5') *applies for the composition of central relations on M.*

Examples: Standard cases, only. In the simplest spaces, we provide the expressions for Δ and analyse the specific nontrivial restrictions which apply, in each case. First, on $\mathbb{R}^2$:

$$\Delta(\vec{x}, \vec{x}_1, \vec{x}_2) = 2\{\vec{x} \wedge \vec{x}_1 + \vec{x}_1 \wedge \vec{x}_2 + \vec{x}_2 \wedge \vec{x}\} \tag{6.6}$$

is the midpoint triangular area, defining via (6.5) the composition of central actions on $\mathbb{R}^2$.

On $\mathcal{T}^2$, we add the restriction on the lengths of the triangles, in terms of midpoints:

$$|q_i - q|, |p_i - p|, |q_1 - q_2|, |p_1 - p_2| < \frac{\pi}{2}, \quad i = 1, 2. \tag{6.7}$$

On the 2-sphere, the midpoint triangular area is given by (see Appendix A):

$$\Delta = 2\,\mathrm{Arg}\left\{\sigma\sqrt{1 - D^2} + \mathrm{i}D\right\}, \tag{6.8}$$

where $\sigma \equiv \sigma(m, m_1, m_2) \in \{-1, 0, 1\}$ has the same sign as any of the scalar products $(\hat{m} \cdot \hat{m}_1)$, $(\hat{m} \cdot \hat{m}_2)$, $(\hat{m}_1 \cdot \hat{m}_2)$, and $D \equiv D(m, m_1, m_2) = \det[\hat{m}, \hat{m}_1, \hat{m}_2]$. Here, $\hat{m} \in S^2 \subset \mathbb{R}^3$. For small triangles, i.e. $|\Delta| < \pi$, (6.8) simplifies to $\Delta(m, m_1, m_2) = 2S^{-1}(\det[\hat{m}, \hat{m}_1, \hat{m}_2])$.

This explicit form for Δ must be placed in (6.5), provided the triplet (m, m_1, m_2) stand as midpoints of a spherical triangle with all sides shorter than π. This nontrivial restriction:

$$\mathrm{sign}(\hat{m} \cdot \hat{m}_1) = \mathrm{sign}(\hat{m}_1 \cdot \hat{m}_2) = \mathrm{sign}(\hat{m}_2 \cdot \hat{m}) \tag{6.9}$$

is the constraint that must be imposed on (6.5) and (6.8) to define the standard composition of two central actions on the sphere. Constraints analogous to (6.9) or (6.7) apply to each mixed composition, for every space with closed geodesics.

The hyperbolic plane presents one subtle distinction. Here, for $m \simeq \vec{m} \in H^2 \subset \mathbb{R}^3$,

$$\Delta(m, m_1, m_2) = 2S^{-1}(\det[\vec{m}, \vec{m}_1, \vec{m}_2]). \tag{6.10}$$

But now we notice that (6.10) only makes sense on the subset of $H^2 \times H^2 \times H^2$ determined by:

$$|\det[\vec{m}, \vec{m}_1, \vec{m}_2]| < 1. \tag{6.11}$$

The composition of two central actions on H^2 is given by (6.5) and (6.10), subject to constraint (6.11). Actually, (6.11) is a necessary and sufficient condition any triplet of points on H^2 must satisfy in order to be the set of midpoints of a hyperbolic triangle.

As in the case of the sphere, there is a nontrivial constraint in the composition of two central actions, but in the hyperbolic case, (6.11) comes naturally from the definition of the area function (6.10), being intrinsic to the hyperbolic geometry. Its interpretation is that the three midpoints can never be too far apart, as can be pictured by drawing a geodesic triangle on the Poincaré disc. For the sphere, on the other hand, (6.9) is a consequence of the restricted groupoid $(TS^2)_0$. In other words, there do exist spherical triangles whose midpoints do not satisfy (6.9), but they cannot be used in the definition of the standard composition of central actions (they can be used for mixed compositions), contrary to the hyperbolic case where, for every triangle, (6.11) holds.

Also note that, when defining triangles by their triplets of vertices, instead of midpoints, no nontrivial restriction exists (except for sets of measure zero), which is a consequence of the fact that the map {vertices} $\to$ {midpoints} is generally less trivial than one would guess by looking only at the euclidean plane. See also Appendix A and [31,33,38,47].

7. Multiple compositions

So far we have seen how two central actions may compose into a new one via (6.5'). Now, we want to generalize it for the composition of an arbitrary number of standard central actions, whenever possible. This should be done with some care, so we first get the following lemma.

Lemma 7.1. *Let f_{α_i} locally be standard central actions for $\Lambda_{\alpha_i} \subset (TU)_0$. If $f_{\alpha_1} \triangle f_{\alpha_2}$ and $f_{\alpha_2} \triangle f_{\alpha_3}$ are unique standard central actions on U, as well as $(f_{\alpha_1} \triangle f_{\alpha_2}) \triangle f_{\alpha_3}$ and $f_{\alpha_1} \triangle (f_{\alpha_2} \triangle f_{\alpha_3})$, then $(f_{\alpha_1} \triangle f_{\alpha_2}) \triangle f_{\alpha_3} = f_{\alpha_1} \triangle (f_{\alpha_2} \triangle f_{\alpha_3})$ is given by*

$$f_{\alpha_1} \triangle f_{\alpha_2} \triangle f_{\alpha_3}(m) = \mathrm{Stat}_{(m_1,m_2,m_3)} \left\{ \sum_i f_{\alpha_i}(m_i) + P_4(m, m_1, m_2, m_3) \right\}, \tag{7.1}$$

P_4 being the smallest symplectic area of any (oriented) quadrilateral decomposable in triangles defined by elements in $(TM)_0^2$ only and determined by the midpoints (m, m_1, m_2, m_3), up to constants.

Proof. Suppose $f_{\alpha_1} \triangle f_{\alpha_2}$, given by (6.5), is a unique standard central action on $U \subset M$, just as f_{α_3}. Applying (6.5) again:

$$\begin{aligned}(f_{\alpha_1} \triangle f_{\alpha_2}) \triangle f_{\alpha_3}(m) &= \mathrm{Stat}_{(m',m_3)}\{f_{\alpha_1} \triangle f_{\alpha_2}(m') + f_{\alpha_3}(m_3) + \Delta(m, m', m_3)\} \\ &= \mathrm{Stat}_{(m',m_3)}\{\mathrm{Stat}_{(m_1,m_2)}\{f_{\alpha_1}(m_1) + f_{\alpha_2}(m_2) \\ &\qquad + \Delta(m', m_1, m_2)\} + f_{\alpha_3}(m_3) + \Delta(m', m_3, m)\}.\end{aligned}$$

If a unique solution exists, this rewrites as:

$$\begin{aligned}&\mathrm{Stat}_{(m_1,m_2,m_3)}\{f_{\alpha_1}(m_1) + f_{\alpha_2}(m_2) + f_{\alpha_3}(m_3) \\ &\quad + \mathrm{Stat}_{(m')}\{\Delta(m', m_1, m_2)\Delta(m', m_3, m)\}\}.\end{aligned}$$

But, with $\{m_i\}$ constrained by the overall stationary condition then, via the central equation (4.3), $\Delta(m', m_1, m_2) =: g_1(m')$ and $\Delta(m', m_3, m) =: g_2(m')$ provide well-defined maps $\vec{G}_1, \vec{G}_2 : m' \mapsto \vec{v}_1, \vec{v}_2 \in (T_{m'}M)_0$, supposing that both g_1 and g_2 , as well as $f_{\alpha_1} \triangle f_{\alpha_2}$ and $(f_{\alpha_1} \triangle f_{\alpha_2}) \triangle f_{\alpha_3}$ are well defined unique central actions, i.e. both partial and complete unique solutions to the stationary conditions exist.

From the form of the central equation and the involutive character of the central potential: $Z_0(\tau) = -Z_0(\bar{\tau})$, the m' stationary condition implies $\vec{G}_1(m') = -\vec{G}_2(m')$.

Using the symmetric exponential map, we see that the two triangles composed form a single quadrilateral, i.e. their sides centered at m' are precisely opposite to each other. That is,

$$\mathrm{Stat}_{(m')}\{\Delta(m', m_1, m_2) + \Delta(m', m_3, m)\} \equiv P_4(m_1, m_2, m_3, m), \tag{7.2}$$

where $P_4(m_1, m_2, m_3, m)$ is the symplectic area of a standard quadrilateral with these midpoints.

Generically, the four midpoints do not determine the quadrilateral uniquely. But if all triangular decompositions yield triangles which are defined by elements in $(TM)_0^2$ only, then by Proposition 6.1 and (7.2), P_4 is the symplectic area of any such quadrilateral, up to constants: any degeneracy in its specific geometry, which is a continuous function of m', does not alter the symplectic area, and any other "$(TM)_0$-quadrilateral", in the above sense, has the same symplectic area up to constants. □

The existence and uniqueness requirements are quite difficult to assure beforehand, in general. Of course, if any f_{α_i} or intermediary composition is not a central action, the triple composition is void. On the other hand, if an intermediary composition is not unique, say $f_{\alpha_1} \triangle f_{\alpha_2} = \{g_1, g_2\}$, then we could proceed to $\{g_1 \triangle f_{\alpha_3}, g_2 \triangle f_{\alpha_3}\}$ in just the same manner, but the final composition $(f_{\alpha_1} \triangle f_{\alpha_2}) \triangle f_{\alpha_3}$ is not guaranteed to be associative, in principle. However, if all $\Lambda_{\alpha_i}, \Lambda_{\alpha_i\alpha_j}, \ldots$ are composed of sheets whose one of them can be consistently singled out, for some particular reason, as well as their corresponding central actions, then we can apply Lemma 7.1 exclusively to this particular set. Carefully reiterating all steps to (7.1) gives the following corollary.

Corollary 7.1. *Let f_{α_i} locally be standard central actions for $\Lambda_{\alpha_i} \subset (TU)_0$. If all intermediary, ordered compositions $f_{\alpha_i} \triangle f_{\alpha_{i+1}}$, $f_{\alpha_i} \triangle (f_{\alpha_{i+1}} \triangle f_{\alpha_{i+2}})$, $((f_{\alpha_i} \triangle f_{\alpha_{i+1}}) \triangle f_{\alpha_{i+2}}) \triangle f_{\alpha_{i+3}}$, etc. are unique standard central actions on U, then any n-string of ordered compositions which is a unique standard central action equals any other such n-string and is given by*:

$$f_{\alpha_1} \triangle f_{\alpha_2} \triangle \cdots \triangle f_{\alpha_n}(m) = \mathrm{Stat}_{(\{m_i\})}\left\{\sum_i f_{\alpha_i}(m_i) + P_{n+1}(m, \{m_i\})\right\}, \tag{7.3}$$

$P_{n+1}(m, \{m_i\})$ being the smallest symplectic area of any (oriented) $(n+1)$-polygon which can be triangulated by elements defined in $(TM)_0^2$ only and determined by the midpoints $(m, \{m_i\})$.

Again, if uniqueness fails, but a unique set of central actions can be consistently singled out, then we can use Corollary 7.1 for these particular standard central actions exclusively (see Section 8). We should note that, with greater care, the above rules can be generalized to multiple mixed compositions of central actions and, even more generally, central relations whenever meaningful.

Corollary 7.1 generalizes a previous result on $\mathbb{R}^{2n}$ [27]. We remark that some of this discussion on composition of midpoint triangles, in general symmetric symplectic spaces, has been approached independently from the point of view of associativity for star products [31].

Examples: For the composition of three actions, the euclidean plane presents an interesting feature. From (6.6) and (7.2), we get $P_4/2 = \mathrm{Stat}_{(\vec{x}')}\{\vec{x}_1 \wedge \vec{x}_2 + \vec{x}_3 \wedge \vec{x} + \vec{x}' \wedge (\vec{x}_1 - \vec{x}_2 + \vec{x}_3 - \vec{x})\}$, which implies: $\vec{x}_1 - \vec{x}_2 + \vec{x}_3 - \vec{x} = 0$. This means that $(\vec{x}, \{\vec{x}_i\})$ are the vertices of a parallelogram with diagonals $\vec{x}_3 - \vec{x}_1$, $\vec{x} - \vec{x}_2$. But this is true for any quadrilateral in $\mathbb{R}^2$, i.e. their midpoints are vertices of a parallelogram. Conversely, given any parallelogram on

$\mathbb{R}^2$, there exists a continuous family of circumscribed quadrilaterals whose midpoints are the vertices of the given parallelogram.

Such a family can be parameterized by one of the vertices of each circumscribed quadrilateral or, equivalently, by the midpoint of one of its diagonals, as $\vec{x}'$ above. In accordance with (7.2), the symplectic area independs on $\vec{x}'$, being uniquely given as twice the area of the inscribed parallelogram: $P_4(\vec{x}, \vec{x}_1, \vec{x}_2, \vec{x}_3) = 2(\vec{x}_1 \wedge \vec{x}_2 + \vec{x}_3 \wedge \vec{x} = \vec{x} \wedge \vec{x}_1 + \vec{x}_2 \wedge \vec{x}_3)$, which, inserted in (7.1), defines the composition of three central actions on $\mathbb{R}^2$, with $(\vec{x}, \{\vec{x}_i\})$ subject to the parallelogram relation, and under the conditions of Lemma 7.1.

On the torus, the same analysis and results apply, but now subject to the extra (standard groupoid) constraint: $|q - q_i|, |q_i - q_j|, |p - p_i|, |p_i - p_j| < \pi/2, i, j = 1, 2, 3$.

On S^2, however, the above quadrilateral ambiguity is an exception, when considering only those quadrilaterals defined by composing elements in $(TS^2)_0$. Then, as with spherical triangles, a near-bijection between the sets of vertices and midpoints allows for a unique definition of the quadrilateral geometry either way, i.e. each quadrilateral is uniquely determined by its vertices or its midpoints (with a few exceptions) and the only restrictions derive from the restricted groupoid $(TS^2)_0$.

Denoting $C_{ij} = \hat{m}_i \cdot \hat{m}_j \equiv \text{cosine}(\text{distance}(m_i, m_j))$, we can write the midpoint area of convex quadrilaterals which are decomposable in small triangles, i.e. $|\Delta| < \pi$, as $P_4(m_1, m_2, m_3, m_4) = 2\sigma_{1234} \cdot C^{-1}\{C_{12}C_{34} + C_{23}C_{41} - C_{13}C_{24}\}$, where $\sigma_{1234} = \pm 1$ is the orientation of (m_1, m_2, m_3, m_4) and the standard groupoid restrictions on the midpoints of these simplest quadrilaterals now become: $D_{123}, D_{234}, D_{341}, D_{412} > 0$, where $D_{ijk} \equiv \det[\hat{m}_i, \hat{m}_j, \hat{m}_k]$, which, together with (7.1), define this simplest composition of three central actions on S^2, under the existence and uniqueness assumptions as required in Lemma 7.1.

On H^2, the midpoint area function for a convex quadrilateral is given similarly by $P_4(m_1, m_2, m_3, m_4) = 2\sigma_{1234} \cdot C^{-1}\{\tilde{C}_{12}\tilde{C}_{34} + \tilde{C}_{23}\tilde{C}_{41} - \tilde{C}_{13}\tilde{C}_{24}\}$, where $\tilde{C}_{ij} = \cosh(\text{distance}(m_i, m_j))$. Again, the restrictions are intrinsic to the hyperbolic geometry and can be obtained directly from the area function, i.e. the convex set (m_1, m_2, m_3, m_4) must satisfy $|\tilde{C}_{12}\tilde{C}_{34} + \tilde{C}_{23}\tilde{C}_{41} - \tilde{C}_{13}\tilde{C}_{24}| < 1$ and each quadruplet of midpoints satisfying this constraint determines a unique convex hyperbolic quadrilateral and vice versa (with a few exceptions, see below). Using Lemma 7.1, we obtain this simplest composition of three central actions on H^2.

See Appendix A for a more detailed analysis on these quadrilateral geometries.

Besides providing explicit equations for compositions of three central actions, the previous discussions further illustrate some kinds of constraints which the midpoints, or centers (the arguments of the composing actions) are subject to.

In this respect, the euclidean plane presents the feature that, when the number of composing actions is even, no restrictions apply, but when the number is odd, there is a degeneracy in the determination of the 2ℓ-polygon from its midpoints, corresponding to a linear functional restriction $g(\vec{x}, \vec{x}_1, \dots, \vec{x}_{2\ell-1}) = 0$ on the arguments of the composing central actions. For the torus, one must further add the groupoid restrictions.

On the other hand, in the nonflat cases studied, such a degeneracy is an exception and its corresponding extra constraint is not present. When considering only those polygons which can be obtained by iterated (standard) central groupoid compositions, there is a near-bijection between the sets of midpoints and vertices. On H^2, intrinsic restrictions on the midpoints exist, though, which show explicitly in the midpoint area function, whatever the number of composing actions. On S^2, only the (standard) central groupoid restrictions apply.

Another way to view degeneracies and extra constraints is the following: Let $(m, \{m_i\}_n)$ be a candidate for the set of midpoints for an $(n+1)$-polygon on M. Also, for each m_i, let $\mathcal{R}_{m_i} : M \to M$ be the corresponding involution whose fixed point is m_i. Then, the existence of a circumscribed $(n+1)$-polygon to the midpoints $(m, \{m_i\}_n)$ is equivalent to the existence of a fixed point for the symplectomorphism $\mathcal{P}_{n+1} \equiv \mathcal{R}_{m_1} \cdot \mathcal{R}_{m_2} \cdots \mathcal{R}_{m_n} \cdot \mathcal{R}_m : M \to M$.

On $\mathbb{R}^2$, when n is odd $\mathcal{P}_{n+1}$ is a translation, see (2.1). Fixed points exist only when this translation is the identity, in which case every point is fixed. For the sphere, on the other hand, $\mathcal{P}_{n+1} \in \mathrm{SO}(3)$, $\forall n \in \mathbb{N}$, and there is always a fixed point (actually two). However, for H^2, $\mathcal{P}_{n+1} \in \mathrm{SO}(2,1)$ and there may or may not exist a fixed point on H^2, $\forall n \geq 2$, but when there exists, it is unique. The exception, in both cases, is when $\mathcal{P}_{n+1} \equiv 1$ and for these sets of points the corresponding circumscribed polygon is not uniquely defined, but so is its area.

We shall not present here an explicit characterization of multiple compositions, for generic n, in every example. The reader is referred to [25], for the euclidean case. Instead, in the next paragraph we study a particular limit for (7.3), as $n \to \infty$.

8. The central variational principle

We now focus on the relationship between finite and infinitesimal canonical transformations in the central description, i.e., on the relation between finite and infinitesimal central actions.

To this end, consider $(\alpha)_T \equiv \{\alpha_y^{(x)}\}_T$, a continuous sequence of canonical transformations on M for which the following properties hold: $\forall x, y, z, t \in [0, T]$ s.t. $x + y + z = t$, $\alpha_y^{(x)} : M \to M$ satisfies $(\alpha_y^{(x)})^*\omega = \omega$, $\alpha_t \equiv \alpha_t^{(0)} = \alpha_z^{(x+y)}(\alpha_y^{(x)} \cdot \alpha_x^{(0)}) = \alpha_z^{(x+y)} \cdot \alpha_{x+y}^{(0)} = (\alpha_z^{(x+y)} \cdot \alpha_y^{(x)}) \cdot \alpha_x^{(0)} = \alpha_{y+z}^{(x)} \cdot \alpha_x^{(0)}$ and also $\alpha_{t+\varepsilon} = \alpha_\varepsilon^{(t)} \cdot \alpha_t = \alpha_t + \mathrm{o}(\varepsilon)$, as $\varepsilon \to 0$, with $\alpha_\varepsilon^{(t)} \to \mathrm{id} : M \to M$, where $\alpha_\varepsilon^{(t)}$ is the infinitesimal canonical transformation defined by the hamiltonian function $h(t)$, via Hamilton's equation or, equivalently, by the infinitesimal central action $f_{\alpha_\varepsilon^{(t)}} = -\varepsilon h(t)$, via the central equation (4.3). We assume that $h(t)$ is a continuous function of t, but $h(t) \neq h(t')$, in general. In other words, h is a nonautonomous hamiltonian, i.e. a continuous function on $M \times [0, T]$. Accordingly, we denote $h(0) \equiv h$ and, if $h(t) = h$, $\forall t \in [0, T]$, we say that h is autonomous. In this particular case, $h(t) \equiv h$, we have that, for $t \in [-T, T]$, $(\alpha_t)^{-1} \equiv \alpha_{-t}$. But generally, i.e. $h(t) \neq h(t')$, such simplest inversibility relation only applies for very short intervals of time, i.e. $(\alpha_\varepsilon^{(t)})^{-1} = \alpha_{-\varepsilon}^{(t)}$ only in the limit $\varepsilon \to 0$. In other words, the flow of $\{\alpha_t\}_T$ is locally hamiltonian (autonomous), but not globally.

Now, we will seek local central actions for α_t given in terms of $h(t)$. To achieve this goal we shall use the results from the previous paragraph, but, in doing so, we should certify that f_{α_t} exists and is unique, $\forall t' \in [0, t)$, in principle. Actually, both conditions can be relaxed in this particular case, as is shown below. Then, we obtain the following theorem.

Theorem 8.1. *Let $(\alpha)_T$ be a continuous sequence of canonical transformations on M, as above, where as $\varepsilon \to 0$, $\alpha_\varepsilon^{(t)}$ is the infinitesimal canonical transformation generated by the hamiltonian $h(t) \equiv h(m,t)$ continuous in t. Then, wherever the central action for $\alpha_t \equiv \alpha_t^{(0)}$ exists, $f_{\alpha_t}(m) \equiv \Psi_h^t(m) \equiv \Psi_h(m,t)$, it satisfies the Central Variational Principle*:

$$\Psi_h(m,t) = \mathrm{Stat}_{(\nu)} \left\{ -\int_\nu h(m'(t'),t')\,\mathrm{d}t' + \$_\nu\omega \right\}(m,t) \tag{8.1}$$

for a family of continuous paths $\nu : [0,t] \to M$ geodesically centered on m, where, by definition $\{\$_\nu\omega\}(m,t) \equiv [\$_\nu\omega]^t(m)$ is the symplectic area between the curve ν and the geodesic from $\nu(t)$ to $\nu(0)$ centered on m (see Fig. 2). *This area function is well defined, up to constants, provided the geodesic is such that the full closed circuit is reducible. Furthermore, the stationary paths ν solving (8.1) coincide with the classical trajectories on phase space describing the continuous evolution from $\nu(0)$ to $\nu(t)$.*

Proof. We start by considering those central actions which are continuously deformed from the hamiltonian, i.e. $f_{\alpha_t} \to -\varepsilon h$, as $t \simeq \varepsilon \to 0$, continuously on $U \subset M$.

Thus, let $\{\Lambda_{\alpha_t}\}$ be a continuous set of lagrangian submanifolds in $((TM)_0, \Omega_0)$, evolving from the zero section $T^0M \equiv \Lambda_{\alpha_0} \simeq M$, s.t. $\forall\tau \in \Lambda_{\alpha_t}$, $\Phi(\tau) = (m_-, m_+)$ satisfies $m_+ = \alpha_t(m_-)$.

Suppose that over $\bar{U} \subset M$ there are ℓ branches of $\Lambda_n \equiv \Lambda_{\alpha_n}$, where $\alpha_n \equiv \alpha_t$ for $t = t_n \in (0,T]$, generated by ℓ central actions $f_{\alpha_n}^j$, $j = 1,\ldots,\ell$. Generically, $\bar{U}$ is an open neighborhood bounded by the sets of central caustics $\{m_n\}^j$, $j = 1,\ldots,\ell-1$. Further, let $t_k \in (0,t_n)$ be the time when a central catastrophe first appear for $\{\Lambda_{\alpha_t}\}$, i.e. Λ_{α_t} satisfies (4.2) everywhere on M, for $t < t_k$.

Denoting $\{m_k\}$ the set of central caustics for Λ_k, then we can find neighborhoods in M divided by $\{m_k\}$. This is not so clear when codim $(\{m_k\}) > 1$, but following the evolution of these caustics we can find appropriate subdivisions in M. Let us denote by U_k^1 one of such neighborhoods, containing $\bar{U}$. Then $\mathcal{A}_\mathrm{c}^0(\bar{U}) \supset \mathcal{A}_\mathrm{c}^0(U_k^1)$. Let $f_{\alpha_k}^1 \in \mathcal{A}_\mathrm{c}^0(\bar{U})$ be the central action for α_k on $\bar{U} \subset U_k^1 \subset M$.

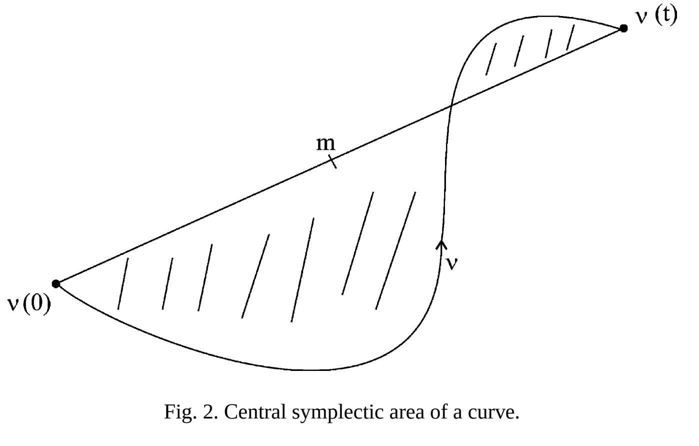

Fig. 2. Central symplectic area of a curve.

Now, of all the branches of Λ_n, over $\bar{U}$, only one is a continuous deformation from Λ_k, over $\bar{U}$, therefore, of all the central actions for α_n on $\bar{U}$, only one is a continuous deformation from $f^1_{\alpha_k}$ on $\bar{U}$. And since $f^1_{\alpha_k}$ is continuously deformed from $-\varepsilon h$, even though there are more than one central actions for α_n on $\bar{U}$, there is a unique central action for α_n on $\bar{U}$, denoted $f^1_{\alpha_n}$, which is continuously deformed from $f_{\alpha_\varepsilon} = -\varepsilon h$, $t \simeq \varepsilon \to 0$.

This result clearly independs on the number of subsets of Λ_n over $\bar{U}$, or equivalently, on the number of central actions for α_n on $\bar{U}$. Also, the other sets of central catastrophes for Λ_n, $\{\tau_n\}^j$, $j = 2, \ldots, \ell - 1$, are not connected to Λ^1_n, so $f^1_{\alpha_n}$ is actually a continuously deformed central action on the larger neighborhood $U^1_n \supset \bar{U}$, $U^1_n \subset U^1_k \subset M$ being limited only by the set of central caustics $\{m_{t'}\}^1$, $\forall t' \in (0, t_n)$. This also means that, for such values of t', $f^1_{\alpha_{t'}}$ exists, besides being continuously deformed from the hamiltonian, on $U^1_n \subset M$.

We are now ready to apply Corollary 7.1 to $f^1_{\alpha_t} = f^1_{\alpha^{(r-1)}_{t/r}} \triangle f^1_{\alpha^{(r-2)}_{t/r}} \cdots \triangle f^1_{\alpha^{(1)}_{t/r}} \triangle f^1_{\alpha^{(0)}_{t/r}}$, r times. That is, the continuity of $f^1_{\alpha_t}$ has substituted for uniqueness. From the involutive character of the central potential Z_0, via the central equation we have that $f_{(\alpha_\varepsilon)^{-1}} = f_{\alpha_{-\varepsilon}} = -f_{\alpha_\varepsilon}$, which implies that $f_{\alpha_{-\varepsilon}} = +\varepsilon h + \mathrm{o}(\varepsilon^3)$. Also for $f_{\alpha^{(t')}_\varepsilon} = -\varepsilon h(t') + \mathrm{o}(\varepsilon^3)$. Then, letting $r \to \infty$, we can safely make the approximation $f^1_{\alpha^{(i)}_{t/r}} = -(t/r)h(t'_i) + \mathrm{o}((t/r)^3) \overset{\sim}{\to} -(t/r)h(t'_i)$, to get on $U^1_t \subset M$:

$$f^1_{\alpha_t}(m) = \lim_{r\to\infty} \left[\mathrm{Stat}_{(\{m'_i\}_r)} \left\{ \sum_{i=1}^{r} \left(\frac{-t}{r} \right) h(m'_i, t'_i) + P_{r+1}(m, \{m'_i\}_r) \right\} \right], \tag{8.2}$$

where $m'_i = m'(t'_i)$, $t'_i < t'_{i+1} \in [0, t]$. Now we realize that (8.2) takes on the form (8.1) as we identify $\nu \equiv \lim_{r\to\infty}(\{m'(t'_i)\}_r) : [0, t] \to U^1_t \subset M$ as a continuous curve satisfying $P_0(\Phi_0^{-1}(\nu(0), \nu(t))) = m$.

In passing from (8.2) to (8.1), this area is integrated via a limit ($r \to \infty$) of the area of $(r + 1)$-polygons whose midpoints are $(m, \{m'_i\}_r)$, as r of the sides tend to length zero while the other tends to the geodesic from m'_r to m'_1 centered on m. In order to see that the stationary path is the classical trajectory with endpoints geodesically centered on m, we notice that each small side of the $(r + 1)$-polygon is a geodesic which, in the limit of very short time intervals, i.e. of very small sides, coincides with the local hamiltonian flow of $h(m'_i, t'_i)$, which is centered on m'_i, as discussed in Section 4. Thus, taking the limit $r \to \infty$ of $\{m'(t'_i)\}_r$, we obtain a path that is everywhere tangent to the locally hamiltonian flow, in other words, that converges onto the classical trajectory.

At first, (8.1) and (8.2) would apply only to those central actions that can be continuously deformed from the hamiltonian h, on $U^1_t \subset M$. However, if f_{α_t} is not of this type, it is always possible to decompose it as $f_{\alpha_t} = f^1_{\alpha^{(t')}_{t-t'}} \triangle f^1_{\alpha^{(0)}_{t'}}$ or $f^1_{\alpha^{(t'+t'')}_{t-t'-t''}} \triangle^j (f^1_{\alpha^{(t')}_{t''}} \triangle f^1_{\alpha^{(0)}_{t'}})$, etc. where each "smaller" component is continuously deformed from the appropriate $h(t')$, on each appropriate neighborhood. Thus, they can be written as solutions to (8.1), with each $\nu' : [t', t - t'] \to M$, etc.

However, the laws for compositions of central actions (6.5'), plus the fact that we are composing central actions for the continuous sequence $(\alpha)_T$, imply that the trajectories

ν', ν'' compose into a single continuous trajectory $\nu = \nu'' \circ \nu'$, for appropriate choices of $f^1_{\alpha_{t'}}$, $f^1_{\alpha^{(t')}_{t''}}$, with the corresponding areas $\mathcal{S}_{\nu'} w$ and $\mathcal{S}_{\nu''} w$ summing up to $\mathcal{S}_\nu w$.

To see this, we notice that the stationary condition on m' in (6.5') implies, via central equation, that $[\mathcal{S}_\nu w]^{t'}(m')$ and $\Delta(m', m'', m)$ provide maps from m' into reciprocally inverse elements in $T_{m'}M$, which, via the symmetric exponential map, tells us that they compose into a single geometric figure. Repeating the analysis with m'', we have that $[\mathcal{S}_\nu w]^{t'}(m')$, $[\mathcal{S}_\nu w]^{t''}(m'')$ and $\Delta(m, m', m'')$ compose into $[\mathcal{S}_\nu w]^t(m) \equiv \{\mathcal{S}_\nu w\}(m, t)$, for $t = t' + t''$, using the fact that $\nu = \nu'' \circ \nu'$, provided the composition is reducible, i.e. provided the triangle of composition $\Delta \equiv \Delta^k_{ij}$ is a reducible circuit, which means that the geodesic from $\nu(t)$ to $\nu(0)$ centered on m is such that it closes the trajectory ν into a reducible circuit. And so on, it follows that f_{α_t} can also be written as a solution to (8.1), even when it is not continuously deformed from the hamiltonian, meaning that, for some $(m', t') \in M \times [0, t)$, $f_{\alpha_{t'}}(m')$ does not exist, i.e. m' is a central caustic singularity for $\Lambda_{\alpha_{t'}}$, $t' < t$.

So, even though generically $f_{\alpha_{t'}}$ does not exist everywhere on M, for $t' < t$, and f_{α_t} is not everywhere deformed from the hamiltonian function h continuously, wherever f_{α_t} exists it can be written as a solution to (8.1), with ν being a continuous trajectory, $\nu : [0, t] \to M$, and m being the center of the (short or long) geodesic from $\nu(t)$ to $\nu(0)$ closing the trajectory into a reducible circuit. $\square$

Theorem 8.1 generalizes to nonflat symmetric symplectic spaces the previous result on euclidean space [25]. It is a real variational principle which is invariant at least under general transformations on M preserving the affine connection and the symplectic form.

Besides, it does not require any local decomposition of the phase space M into lagrangian subsets. In other words, it is fully adapted to the nontrivial geometry of M.

Furthermore, in opposition to the complex counterparts, this real variational principle has only real classical trajectories as stationary solutions.

The novel feature is that the trajectories are constrained on their geodesic centers instead of the more familiar (local) lagrangian coordinates of their endpoints. Therefore, in solving for the paths ν which are stationary in (8.1), only the time t and the center m are held fixed.

Finally, the central action $\Psi_h(m, t)$ provides, via central equation, the finite transformation $\nu(0) \to \nu(t)$ and, given its explicit relationship to the infinitesimal generators $h(m, t)$, plus the fact that it is a real function on $M \times [0, T]$, Ψ_h can be seen as a *finite time extension of the hamiltonian function*.

9. Temporal evolution of central actions

We have just seen how the central variational principle provides, not only for the classical trajectories obtained by the stationary condition, but also for the central actions which generate finite canonical transformations and can thus be seen as finite time extensions of the hamiltonian functions. Now, we shall investigate the temporal evolution of such central actions.

First, we can examine the temporal evolution of $\Psi_h(m,t)$, for fixed m. The total variation of $\Psi_h(m,t)$ with respect to t depends on the direction of $\vec{\dot{m}}$. We have:

$$\varepsilon \cdot \nabla_t(\Psi_h(m,t)) \equiv \varepsilon \cdot \left\{ \frac{\partial \Psi_h(m,t)}{\partial t} + \vec{\dot{m}} \rfloor \mathrm{d}\Psi_h^t(m) \right\}, \tag{9.1}$$

denoting $\delta t = \varepsilon$. On the other hand, by (8.1),

$$\varepsilon \cdot \nabla_t(\Psi_h^t(m)) = -\varepsilon \cdot h(\nu(t),t) + \varepsilon \cdot \{\vec{\dot{m}} \rfloor \mathrm{d}[\mathcal{S}_\nu w]^t(m)\} + \mathrm{o}(\varepsilon^2), \tag{9.2}$$

since $\Psi_h^t(m)$ is stationary in ν and only terms in $(\delta\nu)^2$ contribute, where $\mathrm{o}((\delta\nu)^2) \sim \mathrm{o}(\varepsilon^2)$.

In Eq. (9.2) we are thus approximating the new path $\nu'_{t+\varepsilon}$ by the old one ν_t, i.e. we consider only infinitesimal variations in the endpoint $\nu(t)$ along the same classical trajectory $\nu : (0, t+\varepsilon) \to M$.

Via central equation, we have $\mathrm{d}\Psi_h^t \equiv \mathrm{d}f_{\alpha_t} : m \mapsto \tilde{\tau}_{\alpha_t}$, while $\mathrm{d}[\mathcal{S}_\nu w]^t : m \mapsto \tilde{\tau}_\nu$. But, by construction, $\tilde{\tau}_{\alpha_t} = \tilde{\tau}_\nu = \tilde{\Phi}_i^{-1}(\nu(0),\nu(t))$ and therefore $\mathrm{d}\Psi_h^t(m) = \mathrm{d}[\mathcal{S}_\nu w]^t(m)$, where now we identify $\Psi_h^t(m) \equiv \Psi_h^i(m,t)$ as a generic central action. From this and (9.1) and (9.2), we identify:

$$\frac{\partial \Psi_h(m,t)}{\partial t} = -h(m_+,t). \tag{9.3}$$

Since $m_+ = \exp_m(+\vec{F}^i_{\alpha_t}(m))$, where $\vec{F}^i_{\alpha_t}(m) \in (\widetilde{T_m M})_i$ is defined by $\mathrm{d}f^i_{\alpha_t} \equiv \mathrm{d}\Psi_h^t$ via the central equation, then

$$\mathbb{H}[\Psi_h^i(m,t)] := h(\exp_m(+\vec{F}^i_{\alpha_t}(m)),t) \tag{9.4}$$

defines the functional $\mathbb{H}$ on $\mathcal{A}_\mathrm{c}(U)$. Thus, (9.3) can be rewritten as:

$$\frac{\partial \Psi_h(m,t)}{\partial t} + \mathbb{H}[\Psi_h(m,t)] = 0, \tag{9.5}$$

which is the central version of the *Hamilton–Jacobi equation*.

Examples: On $\mathbb{R}^2$ (9.5) becomes $\partial\Psi_h(\vec{x},t)/\partial t + h(\vec{x} - (1/2)J(\partial\Psi_h(\vec{x},t)/\partial\vec{x}),t) = 0$, see [21,27].

On S^2 and H^2, however, its explicit generic form in local coordinates is quite complicated and it is rather simpler to use (4.9) or (4.11) to write $\mathbb{H}[\Psi_h^t(m)]$, for each specific h.

As simplest example, consider on S^2 the hamiltonian function $h = -C_\theta$, generator of infinitesimal rotations around the south/north axis, or poles. Using the convention $\Psi_h^t \equiv f_{\alpha_t} \equiv 2f_t$, by (4.9) we rewrite the standard version of (9.5) as $\partial f(\theta,t)/\partial t - (1/2)C_\theta\sqrt{1-(\partial f(\theta,t)/\partial\theta)^2} = 0$, where we denoted $f_t(\theta,\varphi) \equiv f(\theta,t)$, exploiting the φ-invariance of the action; and remembering that, by rescaling $\Psi_h^t \equiv f_{\alpha_t}$, we must also rescale h, and hence $\mathbb{H}$ by the same factor. Check that the central action given by (5.3), with $\chi = 0$ and $t = 2\gamma$, satisfies the previous equation.

Finally, we notice that we could rewrite Eq. (9.1) as: $\nabla_t^g(\Psi_h^t(m)) := \partial\Psi_h^t(m)/\partial t + \vec{v}_g \rfloor \mathrm{d}\Psi_h^t(m)$, for $\vec{\dot{m}} \equiv \vec{v}_g$ defined by $\vec{v}_g \rfloor w = -\mathrm{d}g$, $g \in \mathcal{C}_R^k(M)$. Using (9.5), we get the following proposition.

Proposition 9.1. *Let* $g, h \in \mathcal{C}^k_R(M \times [0, T])$ and $\Psi^t_h \in \mathcal{A}_c(U)$, $U \subset M$, where $\Psi^t_h \equiv \Psi_h(t)$ *is related to h via the central variational principle* (8.1), *for a given time* $t \in [0, T]$. *The time derivative of* Ψ^t_h *"along g", i.e. in the direction of the local hamiltonian flow of g, denoted* $\nabla^g_t(\Psi^t_h)$, *is given by*

$$\nabla^g_t(\Psi_h(t)) = \{\Psi_h(t), g(t)\} - \mathbb{H}[\Psi_h(t)], \tag{9.6}$$

where $\{\ ,\ \}$ *is the Poisson bracket and the functional* $\mathbb{H}$ *is defined by* (9.4) *via central equation.*

Remark that the RHS of (9.6) involves partial derivatives in M only. Particularly interesting is the case where Ψ^t_h is known to be invariant in a given direction $\vec{v}_g$,

$$\nabla^g_t(\Psi_h(t)) = 0, \quad \text{giving} \quad \{\Psi_h(t), g(t)\} = \mathbb{H}[\Psi_h(t)] \tag{9.7}$$

as a direct relationship between the functional $\mathbb{H}$ and the Poisson bracket with g.

Conversely, any function g satisfying the second part of (9.7) defines curves $\Gamma_g : [0, T] \to U \subset M$, along which Ψ^t_h is constant. The other particularly interesting case is when $\{\Psi_h(t), g(t)\} = 0$, for which $\nabla^g_t(\Psi_h(m, t)) = \partial\Psi_h(m, t)/\partial t$, computable by the Hamilton–Jacobi equation.

10. Conclusion

We have presented a general formalism for describing hamiltonian systems defined on symmetric symplectic spaces, where the local generating functions are real functions on phase space. We have seen how the central actions (relations) are defined, generating finite canonical transformations (relations) via the central equation and the symmetric exponential map, and how they compose via a neat formula involving the midpoint triangular (polygonal) area. We have also seen how the "extended hamiltonians" satisfy a geometrically simple real variational principle, which determines the classical trajectories, and satisfy a Hamilton–Jacobi equation, mixed with Poisson brackets.

The authors' main motivation into this central formalism lies in its application to problems in quantization and semiclassical analysis, more specifically in connection to "Weyl quantization" and "star products", which attempt to understand the classical–quantum relationship within the phase space formalism. In this respect, and specially for oscillatory phenomena, it turns into a definite advantage the definition of real phase space generating functions, which can be connected to hamiltonians in such geometrical fashion, with their neat triangular law of composition (see [5,27,31–33,47]).

A possible extra application refers to implementing new symplectic integrators (see [24] for a review). Here, the polygonal law of composition (7.3) could be applied in the discretization process, making use of the fact that each local action for a finite small interval of time is a small, in principle controlled, deformation of the hamiltonian. Also, for autonomous hamiltonians the formalism is symmetric with respect to trajectories in both temporal directions.

On its own, however, such geometrically simple law of composition presents new routes of investigation on symplectic dynamical systems. For, as hamiltonian functions correspond to generators of infinitesimal canonical transformations, the actions correspond to elements in the Lie group. Therefore, their homogeneous presentation, which naturally extends to relations, sheds new light on the canonical formalism on symmetric symplectic spaces.

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Appendix A

A.1. Triangular computations

In the spherical case, recall [35] that if λ_i are the angles of a geodesic triangle on S^2, whose opposite sides are $\ell_i < \pi$, respectively, then the following trigonometric equalities hold: $S_{\ell_i}/S_{\lambda_i} =$ constant, $C_{\ell_k} = C_{\ell_i}C_{\ell_j} + S_{\ell_i}S_{\ell_j}C_{\lambda_k}$, $C_{\lambda_k} = S_{\lambda_i}S_{\lambda_j}C_{\ell_k} - C_{\lambda_i}C_{\lambda_j}$. Also, the area of a spherical triangle is its excess angle [35].

Now, for any spherical triangle with sides $\ell_i < \pi$, a simple computation shows that $T_{\lambda_i} = K/(C_{\ell_i} - C_{\ell_j}C_{\ell_k})$ where λ_i is the angle opposite to ℓ_i and K is a constant for this triangle, $K^2 \equiv 1 - C^2_{\ell_1} - C^2_{\ell_2} - C^2_{\ell_3} + 2C_{\ell_1}C_{\ell_2}C_{\ell_3} \equiv \mathrm{Det}^2[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3]$, where α_i are the vertices of the triangle. By correctly fixing the orientations, we can take the $+$ sign in the square root.

Let us denote $\ell_i = 2y_i$ and $x_i \equiv \mathrm{distance}(m_j, m_k)$, where m_i is the midpoint of the ℓ_i side. Then $\hat{m}_i = (1/2C_{y_i})(\hat{\alpha}_j + \hat{\alpha}_k)$, from which, since $C_{x_i} = \hat{m}_j \cdot \hat{m}_k$, we get that $C_{x_i} = (1/2C_{y_k})(C_{y_j} + C_{z_j}) = (1/2C_{y_i})(C_{y_k} + C_{z_k})$, where $z_j \equiv \mathrm{distance}(m_j, \alpha_j)$. From this we get: $C_{x_1}/C_{y_1} = C_{x_2}/C_{y_2} = C_{x_3}/C_{y_3} = \Gamma$, a constant. This is a generalization of the plane trigonometric relation $x_i/y_i = 1$.

Now, in order to compute Γ, we substitute the previous equation in the trigonometric equalities, to obtain: $\Gamma^2 = C^2_{x_1} + C^2_{x_2} + C^2_{x_3} - 2C_{x_1}C_{x_2}C_{x_3} \equiv 1 - \mathrm{Det}^2[\hat{m}_1, \hat{m}_2, \hat{m}_3]$. Using the previous equations for T_{λ_i}, Γ and Γ^2, we finally get $T_{\lambda_i} = \Gamma\sqrt{1-\Gamma^2}/(\Gamma^2 - C_{x_j}C_{x_k}/C_{x_i})$ and $T_{(\lambda_1+\lambda_2+\lambda_3)} \equiv T_\Delta = \Gamma\sqrt{1-\Gamma^2}/(\Gamma^2 - 1/2)$. Identifying $\Gamma \equiv C_\gamma$, we have $T_\Delta = T_{2\gamma}$, that is $C_{\Delta/2} = \pm\sqrt{C^2_{x_1} + C^2_{x_2} + C^2_{x_3} - 2C_{x_1}C_{x_2}C_{x_3}} \equiv \Gamma$, $S_{\Delta/2} = \mathrm{Det}[\hat{m}_1, \hat{m}_2, \hat{m}_3]$.

The sign choice for $S_{\Delta/2}$ is fixed by the orientation. We still have to determine the sign of the square root in $C_{\Delta/2}$. Obviously, if $|\Delta| < \pi$, we must choose the $+$sign. These triangular areas are continuously deformed from infinitesimal triangles, for which $C_{x_i} > 0, \forall i$. Since we are considering only short triangles, i.e. $y_i < \pi/2, \forall i$, from $C_{x_i}/C_{y_i} = \Gamma$, we get $\Gamma > 0$.

On the other hand, let $|\Delta| = 2\pi$, i.e. consider m_1, m_2, m_3 to be collinear, same for α_1, α_2, α_3, such that the "triangle" coincides with a geodesic meridian. Again, if $y_i < \pi/2, \forall i$, it is clear that in this case $C_{x_i} < 0, \forall i$, from $C_{x_i}/C_{y_i} = \Gamma$, since $C_{\Delta/2} \equiv \Gamma < 0$. And so on for triangular areas continuously deformed from this "big triangle".

Finally, when $|\Delta| = \pi$, $C_{\Delta/2} = 0$ and we have that, $\forall i$, $C_{x_i} = 0$. In this case, C_{y_i} is completely undetermined and so is the triangle, although the area of all such triangles is uniquely given by their common midpoints. It follows that the sign of the square root is the same as the sign of the C_{x_i}, or in other words $\hat{m}_j \cdot \hat{m}_k$, $\forall i, j, k$, if all sides are short. Hence, we have got (6.8).

As for the restrictions on the midpoints, if $C_{y_i} > 0, \forall i$, then either $C_{x_i} > 0, \forall i$, or $C_{x_i} = 0$, $\forall i$ or else $C_{x_i} < 0$, $\forall i$. In all cases, $\mathrm{sign}(C_{x_i}) = \mathrm{sign}(C_{x_j})$, $\forall i, j$, which is condition (6.9). With some care, these results can be modified for general spherical triangles.

Again, the previous analysis can be adapted to H^2, with some care (refer to [35]). On H^2 we do not have problems of antipodals or sign choices, since $|\Delta| < \pi$, always. On the other hand, the analogous to (6.8), namely (6.10), is well defined only when (6.11) is satisfied. But once $\Gamma = C_{\Delta/2} = \tilde{C}_{x_1} + \tilde{C}_{x_2} + \tilde{C}_{x_3} - 2\tilde{C}_{x_1}\tilde{C}_{x_2}\tilde{C}_{x_3}$ is well defined, we go through the argument backwards from $\tilde{C}_{x_1}/\tilde{C}_{y_1} = \Gamma$, to see that the triangle is also well defined.

See [38] for an alternative discussion on these triangular geometries.

A.2. Quadrilateral computations

Again we proceed in the spherical case and later adapt the hyperbolic formulas. Consider a short quadrilateral with vertices α_i and midpoints $m_i \equiv \mathrm{mid}(\alpha_i, \alpha_{i+1})$, s.t. $\mathrm{distance}(\alpha_i, \alpha_j) < \pi$, $\forall i, j \in \{1, \ldots, 4\}$. This means that not only the sides, but also the diagonals are short. Now, denote by m_0 the midpoint of the diagonal (α_1, α_3) and by y_0 its half length. Similarly, denote by $y_i = (1/2)\mathrm{distance}(\alpha_i, \alpha_{i+1})$. Then, each of the triangles $(\alpha_1, \alpha_2, \alpha_3)$ and $(\alpha_3, \alpha_4, \alpha_1)$ are uniquely determined by their midpoints (m_0, m_1, m_2) and (m_0, m_3, m_4), which shall be denoted by Δ_{12} and Δ_{34}, respectively, with the same notation referring to their respective areas. Further, we denote $x_{ij} = \mathrm{distance}(m_i, m_j)$.

From the triangular analysis, we know that $C_{x_{12}}/C_{y_0} = C_{\Delta_{12}/2}$, $C_{x_{34}}/C_{y_0} = C_{\Delta_{34}/2}$. Similarly for the other partition, $C_{x_{23}}/C_{y_0'} = C_{\Delta_{23}/2}$, $C_{x_{41}}/C_{y_0'} = C_{\Delta_{41}/2}$, where $y_0' = (1/2)$ distance (α_2, α_4), m_0' being its midpoint, and so on. Therefore, $C_{x_{12}}/C_{x_{34}} = C_{\Delta_{12}/2}/C_{\Delta_{34}/2}$, $C_{x_{23}}/C_{x_{41}} = C_{\Delta_{23}/2}/C_{\Delta_{41}/2}$. These equations generalize the parallelogram relation on the plane, $x_{12}/x_{34} = 1 = x_{23}/x_{41}$, but contrary to the plane, they impose no constraint on the midpoints.

In compensation, on S^2 the four midpoints uniquely determine the area as well as the specific geometry of the short quadrilateral. That is, $m_0 = m_0(\{m_i\})$ and similarly for m_0'.

To see this, let us denote $\tau_0 = \Phi_0^{-1}(\alpha_1, \alpha_3)$, i.e. $\tau_0 = (m_0, \vec{v}_0)$, $|\vec{v}_0| = y_0$. Also, we denote by $\mathcal{R}_m$ the involution through m and consider the element of SO(3) defined as $\sigma_{12}^2 := \mathcal{R}_{m_1}\mathcal{R}_{m_2}$. Similarly, $\sigma_{34}^2 := \mathcal{R}_{m_3}\mathcal{R}_{m_4}$. Now, for any element $\sigma \in \mathrm{SO}(3)$, consider the vector field $X_\sigma \subset TS^2$ defined by $\tau \in X_\sigma$ iff $\Phi(\tau) = (m_-, m_+)$ s.t. $m_+ = \sigma^2(m_-)$.

The condition which guarantees that triangles Δ_{12} and Δ_{34} compose into a quadrilateral $\Box_{1234}$ can be written as $\tau_0 \in X_{\sigma_{12}} \cap X_{\sigma_{34}^{-1}}$. But, $\tau_0 \in X_{\sigma_{12}}$ only if the pole p_{12} of σ_{12} lies in the polar line of τ_0, which is defined as the orthogonal geodesic to $\vec{v}_0$, at m_0. Thus, the first condition obtained is that m_0, p_{12} and p_{34} be collinear. That is, $\mathrm{Det}[\hat{m}_0, \hat{p}_{12}, \hat{p}_{34}] = 0$.

In fact, we have more: $\hat{p}_{12} = (1/S_{x_{12}})\hat{m}_1 \times \hat{m}_2$, $\hat{p}_{34} = (1/S_{x_{34}})\hat{m}_3 \times \hat{m}_4$. Therefore, $\mathrm{Det}[\hat{m}_0, \hat{m}_1 \times \hat{m}_2, \hat{m}_3 \times \hat{m}_4] = 0$. We can thus write $\hat{m}_0 = \vec{z}/|\vec{z}|$, where $\vec{z} = z_{12}(\hat{m}_1 \times \hat{m}_2) + z_{34}(\hat{m}_3 \times \hat{m}_4)$. Further, denoting $\mu_{12} = \mathrm{distance}(m_0, p_{12})$ and $\mu_{34} = \mathrm{distance}(m_0, p_{34})$, we have $|S_{\mu_{12}}T_{x_{12}}| = |T_{y_0}| = |S_{\mu_{34}}T_{x_{34}}|$, but since $|S_{\mu_{12}}S_{x_{12}}| = |\hat{m}_0 \times (\hat{m}_1 \times \hat{m}_2)|$ and $|S_{\mu_{34}}S_{x_{34}}| = |\hat{m}_0 \times (\hat{m}_3 \times \hat{m}_4)|$, it follows that $|\hat{m}_0 \times (\hat{m}_1 \times \hat{m}_2)| \cdot |\hat{m}_3 \cdot \hat{m}_4| = |\hat{m}_0 \times (\hat{m}_3 \times \hat{m}_4)| \cdot |\hat{m}_1 \cdot \hat{m}_2|$. Then, we get $\hat{m}_0 = \vec{z}/|\vec{z}|$, where $\vec{z} = \alpha C_{x_{12}}(\hat{m}_3 \times \hat{m}_4) + \beta C_{x_{34}}(\hat{m}_1 \times \hat{m}_2)$, $\alpha, \beta \in \{-1, 1\}$.

The sign choices must be made with care, but for small convex quadrilaterals we take the $+$ choice twice. In this case, with $R = C_{x_{12}}C_{x_{34}} + C_{x_{23}}C_{x_{41}} - C_{x_{13}}C_{x_{24}}$, we have that $|\vec{z}|^2 \equiv z^2 = C_{x_{12}}^2 + C_{x_{34}}^2 - 2C_{x_{12}}C_{x_{34}}R$. Analogous equations hold for $\hat{m}_0'$.

We have fixed the geometry of the small convex quadrilateral uniquely from the midpoints $\{m_1, \dots, m_4\}$, as mentioned earlier. The exception is when $\mathcal{R}_1\mathcal{R}_2\mathcal{R}_3\mathcal{R}_4 = 1$, but now, contrary to the euclidean plane, this is really the exception, not the rule.

As for the area, from (6.8) we have $S_{\Delta_{12}/2} = (1/z)\{C_{x_{34}} - RC_{x_{12}}\}$, $S_{\Delta_{34}/2} = (1/z)\{C_{x_{12}} - RC_{x_{34}}\}$. Then, in this simpler case, denoting the quadrilateral area by P_4, we get $C_{P_4/2} = R$, $P_4 = 2\sigma_{1234} \cdot C^{-1}\{C_{x_{12}}C_{x_{34}} + C_{x_{23}}C_{x_{41}} - C_{x_{13}}C_{x_{24}}\}$.

Here we have restricted to convex quadrilaterals decomposed in small triangles (area $< \pi$). Greater care is needed for the sign choices in the expression of $\hat{m}_0$, otherwise. As for the restrictions, in this simpler case, by imposing $\hat{m}_0 \cdot \hat{m}_i > 0$, we get $D_{123} > 0$, and cyclic.

Again, in transposing to the hyperbolic plane, we do not have as many worries about sign choices, however, in this case the analogous area function, $P_4 = 2\sigma_{1234} \cdot C^{-1}(\tilde{R})$, where $\tilde{R} = \tilde{C}_{12}\tilde{C}_{34} + \tilde{C}_{23}\tilde{C}_{41} - \tilde{C}_{13}\tilde{C}_{24}$, is well defined only when $|\tilde{R}| < 1$ and, as in the triangular analysis, once this holds the convex quadrilateral exists. And so on for the general case.

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